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MDCCCXII.

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THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the council-books and journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries, till the Forty-seventh Volume: the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to

be, the importance and singularity of the subjects, or the advantageous manner of treating them, without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they receive them, are to be considered in no other light than as a matter of civility, in return for the respect shewn to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped, that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

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APPENDIX.

PHILOSOPHICAL

TRANSACTIONS.

I. *On the Grounds of the Method which* Laplace *has given in the second Chapter of the third Book of his* Mécanique Céleste *for computing the Attractions of Spheroids of every Description. By* James Ivory, *A. M. Communicated by* Henry Brougham, *Esq. F. R. S. M. P.*

Read July 4, 1811.

In every physical inquiry the fundamental conditions should be such as are supplied by observation. Were it possible to observe this rule in every case, theory would always comprehend in its determinations a true account of the phenomena of nature. Applying the maxim we have just mentioned to the question concerning the figure of the planets, the mathematician would have to investigate the figure which a fluid, covering a solid body of any given shape, and composed of parts that vary in their densities according to a given law, would assume by the joint effect of the attraction on every particle and a centrifugal force produced by a rotatory motion about an axis. The circumstances here enumerated are all that observation fully warrants us to adopt as the foundation of

this inquiry: for, with regard to the earth we know little more than that it consists of a solid nucleus, or central part, covered with the sea; and with regard to the other planets, all our knowledge is derived from analogy which leads us to think that they are bodies resembling the earth. There is one consideration, however, by which the general research may be modified without hurting the strictest rules of philosophizing; and that is, the near approach to the spherical figure which is observed in all the celestial bodies: and it is fortunate that this circumstance contributes much to lessen the great difficulties that occur in the investigation. But, even with the advantage derived from this limitation, the inquiry is extremely difficult, and leads to calculations of the most abstruse and complicated nature; and, when viewed in the general manner we have mentioned, it far surpassed the power of the mathematical and mechanical sciences as they were known in the days of Sir Isaac Newton, who first considered the physical causes of the figure of the planets. That great man was therefore forced to take a more confined view of the subject, and to admit such suppositions as seemed best adapted to simplify the investigation. He supposed in effect that the earth and planets at their creation were entirely fluid, and that they now preserve the same figures which they assumed in their primitive condition; a hypothesis by which the inquiry was reduced to determine the figure necessary for the equilibrium of a fluid mass. The mathematicians, who have followed in the same tract of inquiry, have seldom ventured to go beyond the limited supposition proposed by Newton. They have succeeded in shewing that a mass revolving about an axis, and composed of one fluid of a uniform density, or

of different fluids of different densities, will be in equilibrium, and will for ever preserve its figure when it has the form of an elliptical spheroid of revolution oblate at the poles. It has likewise been proved that the same form is the only one capable of fulfilling the required conditions; which completes the solution of the problem in so far as it regards a mass entirely fluid.

The hypothesis of NEWTON, although most judicious, and best adapted for simplifying the investigation, is nevertheless quite arbitrary, and indeed does not seem to agree well with what is observed at the surface of the earth. Had the terrestrial globe been once entirely fluid, the heterogeneous matters of which it consists, must have taken an arrangement depending on their densities; the substances of greatest density would ultimately have settled at the centre, and those of least density at the surface; and in proceeding from the centre to the surface, the changes of density would not have been very sudden, but slow and gradual and hardly perceptible for considerable depths. Admitting this hypothesis we should therefore expect to find all the matter at the earth's surface, or near it, little different in respect of density; which is quite contrary to experience, since nothing can be more unequal and irregular than the density of the substances that compose the upper strata of the earth. Many other phenomena are also inconsistent with that uniform arrangement of parts which seems to be a necessary consequence of the supposition that the earth was originally fluid: of this description are, the great elevation of the continents above the surface of the sea; the depth of the immense channels which contain the waters diffused over the surface of the globe; and the irregular

disposition of the land and water on the same surface. Besides all this, after a long discussion, in which every circumstance that can affect the question, has been duly weighed, it seems now to be ascertained, that the elliptical figure of the earth, cannot be reconciled with the actual measurements which have been made for the express purpose of bringing the theory to the test of experiment. The hypothesis of Newton is therefore not exactly consonant to observation: and we must infer that the solid part of the earth is not, at least in the present state of the globe, possessed of that regularity of figure, nor of that peculiar disposition of the internal strata, which would arise from the earth's having been originally fluid. Hence it becomes necessary to consider the question of the figure of the planets in a more enlarged point of view; to free it from all arbitrary suppositions, and to attempt such a solution of the problem, as shall apply to whatever figure or hypothesis may appear most agreeable to observation. It is in this way only, that theory and observation can mutually assist one another, and ultimately lead us to the truth—that theory can prompt observation, and observation perfect and confirm theory.

The celebrated French mathematician, D'Alembert, was the first who contemplated the question of the figure of the planets in a general manner, by extending his researches to other figures than the elliptical spheroid. The difficulty is to investigate the attractive force of a body of any proposed figure, and composed of strata that vary in their densities, according to any given law. D'Alembert invented a method for this purpose which, although it is very ingenious, and so general as to apply in a great variety of cases, is nevertheless

destitute of that simplicity which is absolutely necessary for advancing our knowledge in an enquiry so complicated in all respects.

LAPLACE, to whom every part of physical astronomy owes so much, has been very successful in improving that branch of it which relates to the figure of the planets, and to other questions with which this is connected. The foundation of his researches on this subject, is laid in the second chapter of the third book of the *Mécanique Céleste*, where he treats of the attractions of spheroids in general, and more particularly of such as differ but little from spheres. The investigation required in this part of physics, if it be guided by the desire of obtaining useful conclusions, is not only extremely difficult, but of a nature so nice and delicate, as would at first seem to elude the ordinary methods of analysis, and to require particular contrivances adapted to the exigencies of the case. When a fluid covering a solid body, has assumed a permanent figure, that figure will depend upon the gravity at the surface; while the same gravity, being the combined effect of the attractions of all the molecules of the compound body, is itself produced by the form of the surface. Thus the figure of the surface is in a manner both a *datum* and *quæsitum* of the problem; and the skill of the analyst must be directed to find an expression of the intensity of the attractive force which shall be sufficiently simple, and shall likewise preserve in it the elements of the figure of the attracting solid. All these conditions are fulfilled in the skilful solution of the problem of attractions given by LAPLACE, in which the relation between the radius of the spheroid and the series for the attractive force on a point without, or within, the surface, or on it, is

deduced in a manner admirably simple, when the complicated nature of the question is considered.

In order to give a succinct view of the plan of analysis pursued by Laplace, we must begin with observing that he does not seek directly an expression of the attractive force, but that he investigates the value of another function from which the attractive force in any proposed direction, may be derived by easy algebraic operations. This function, which in the law of attraction that obtains in nature, is the sum of all the molecules of the attracting solid, divided by their respective distances from the attracted point, he expands in all cases into a series, containing the descending powers of the distance of the attracted point from the center, when that point is without the surface; but the ascending powers of the same distance, when the attracted point is within the surface: and the question is, to determine the coefficients of the several terms of the expansion. In the first place, it is proved that every one of the coefficients satisfies an equation in partial fluxions, first noticed by the author himself, and from the skilful use of which, all the advantages peculiar to his method are derived. Laplace next lays down a theorem, which, he affirms, is true at the surfaces of all spheroids that differ but little from spheres; hence he deduces the value of an expression, which is the sum of all the coefficients sought respectively multiplied by a known number; and, what is remarkable, the value alluded to, is found to be proportional to the difference between that radius of the spheroid which is drawn through the attracted point, and the radius of the sphere nearly equal to the spheroid. The circumstances we have now mentioned, suggest an elegant solution of the problem, and one that has

the advantage of expressing the radius of the spheroid and the series for the attractive force, by means of the same functions. For in order to find the coefficients sought, we have only to develope the difference between the radius of the spheroid, and the radius of the sphere, into a series of parts, every one of which shall satisfy the equation in partial fluxions: and LAPLACE not only gives a method for computing the several parts, but he likewise proves that the developement is unique, or can be made no more ways than one.

The solution, of which we have endeavoured to give a concise notion, is not more important for the physical consequences which flow from it, than it is curious in an analytical point of view, for the singular art with which the author has avoided the complicated integrations that naturally occur in the investigation, and has substituted in their room the easy operations of the direct method of fluxions. He has been enabled to do this by the help of the theorem which he had discovered to be true at the surfaces of all spheroids that nearly approach the spherical figure. In the *Mécanique Céleste*, the proposition just mentioned is enunciated in the most general manner, comprehending every case in which the attractive force is proportional to any power of the distance between the attracting particles :* but in order to avoid every discussion not essential to the main scope of this discourse, I shall chiefly confine my attention to the case of nature in which the attraction follows the inverse proportion of the square of the distance ;† this being the only case which it is really interesting to consider, because it is the only one that enters into the inquiry concerning the figure of the planets. The

* Liv. 3. No. 10. Equat. (1)

† Ib. Equat. (2)

theorem, it may be remarked, is merely laid down by the author, and the truth of it confirmed by a demonstration; it does not arise naturally in the course of the analysis; and the reader of the *Mécanique Céleste* is at a loss to conjecture by what train of thought it may have been originally suggested. It may be doubted whether the theorem was introduced for the sake of demonstrating a method of investigation previously known to be just from other principles; or whether it preceded in the order of invention, and led to the method of investigation. But however this may be: after having studied the part of LAPLACE's work referred to with all the attention which the importance of the subject and the novelty of the analysis both conspire to excite, I cannot grant that the demonstration which he has given of his proposition is conclusive. It is defective and erroneous, because a part of the analytical expression is omitted without examination, and rejected as evanescent in all cases; whereas it is so only in particular spheroids, and not in any case on account of any thing which the author proves. Two consequences have resulted from this error; for, in the first place, the method for the attraction of spheroids, as it now stands in the *Mécanique Céleste*, being grounded on the theorem, is unsupported by any demonstrative proof; and, secondly, that method is represented as applicable to all spheroids differing but little from spheres, whereas it is true of such only as have their radii expressed by functions of a particular class.

In a work of so great extent as the *Mécanique Céleste*, which treats of so great a variety of subjects, all of them very difficult and abstruse, it can hardly be expected that no slips nor inadvertencies have been admitted. On the other

hand, the genius of the author is so far above the ordinary cast; his knowledge of the subjects he treats is so profound; and the correctness of his views is established by so many important discoveries, that so high an authority is not to be contradicted on any material point without the greatest caution and on the best grounds. It is also to be observed that the *Mécanique Céleste* has now been many years before the public: and although the problem of attractions is the foundation of many important researches, and is more particularly recommended to the notice of mathematicians by the novelty and uncommon turn of the analysis; on which account it may be supposed to have been scrutinized with more than an ordinary degree of curiosity; yet nobody has hitherto called in question the accuracy of the investigation. These considerations will no doubt occasion whatever is contrary to the doctrines of LAPLACE, and more especially to his theory of the attractions of spheroids, to be received with some degree of scepticism: they ought certainly to do so; but our respect even for his authority ought not to be carried so far, as to preclude all criticism of his works, or dissent from his opinions. The writings of no author on any subject deserve to have more respect and deference paid to them, than the writings of LAPLACE on the subject of physical astronomy; with this no one can be more deeply impressed than the author of this discourse; and it was not till after much meditation that, yielding to the force of the proofs which are now to be detailed, he has ventured to advance any thing in opposition to the highest authority, in regard to mathematical and physical subjects, that is to be found in the present times.

1.* Conceive a spheroid which differs but little from a sphere, and also a point or centre in the middle; let ρ denote the radius of the spheroid drawn to an attracted point in the surface: then the whole spheroid will consist of two parts, viz. a sphere of which the radius is ρ, and a shell of matter spread over the surface of the sphere every where so thin as to contain only one molecule in the depth. The function V (which, in the law of attraction that takes place in nature, is the sum of all the molecules of the attracting body divided by their respective distances from the attracted point), relatively to the whole spheroid, will be determined by seeking its value, 1st. relatively to the sphere; 2dly, relatively to the shell of matter.

Produce the radius ρ without the surfaces of the spheroid and sphere, till the distance from the centre be r; then the value of V, relatively to the sphere, for the attracted point situate at the extremity of r, will be $\frac{4\pi}{3} \cdot \frac{\rho^3}{r}$† ($\pi$ denoting the periphery when the diameter is unit); and, making $r = \rho$, it will be $\frac{4\pi}{3} \cdot \rho^2$, for the point in the surface at the extremity of ρ. Again, let dm be one of the indefinitely small molecules in the difference between the spheroid and the sphere; and let f denote the distance of the same molecule from the attracted point in the surface at the extremity of ρ; then the value of V, relatively to the shell of matter spread over the surface of the sphere will be $= \int \frac{dm}{f}$, the fluent being extended to all the molecules in the shell, those on the outside of

* Méc. Céleste, Liv. 3, No. 10. † Liv. 2d, No. 12.

the sphere being positive, and those on the inside negative. Therefore, relatively to the whole spheroid, we shall have

$$V = \frac{4\pi}{3} \cdot \rho^2 + \int \frac{dm}{f}. \quad (A)$$

We must next compute the value of $\left(\frac{dV}{dr}\right)$ in the same circumstances as before. Relatively to the sphere, it is $-\frac{4\pi}{3} \cdot \frac{\rho^3}{r^2}$ for the point without the surface: and, by making $r = \rho$, it is $-\frac{4\pi}{3} \cdot \rho$ for the point in the surface. In order to find the other part of the quantity in question we may suppose, with LAPLACE,* the attracted point to be raised up, in the prolongation of ρ, the distance δr above the surfaces of the spheroid and sphere; then, if f' denote the distance of the molecule dm from the attracted point in its new position, $\int \frac{dm}{f}$ and $\int \frac{dm}{f'}$ will be two consecutive values of the same function which correspond to the values r and $r + \delta r$; therefore, supposing r to vary, the fluxional coefficient will, by the principles of the differential calculus, be $= \frac{\int \frac{dm}{f'} - \int \frac{dm}{f}}{\delta r}$ when $\delta r = o$. Therefore, by adding together the two parts of $\left(\frac{dV}{dr}\right)$, we shall get

$$\left(\frac{dV}{dr}\right) = -\frac{4\pi}{3} \cdot \rho + \frac{\int \frac{dm}{f'} - \int \frac{dm}{f}}{\delta r}; \quad (B)$$

observing that the second term on the right-hand side is to be valued on the supposition of $\delta r = o$.

Let γ denote the cosine of the angle contained by ρ and another radius of the sphere drawn to the molecule dm; then f, the distance of the molecule from the attracted point in the

* Liv. 3e, No. 10.

first position, will be $= \rho\sqrt{2(1-\gamma)}$; and f', the same distance in the second position, will be $= \left\{(\rho+\delta r)^2 - 2\rho(\rho+\delta r)\cdot\gamma + \rho^2\right\}^{\frac{1}{2}}$; and if, with Laplace, we neglect the square and other higher powers of δr, then $f' = \left\{1 + \frac{1}{2}\cdot\frac{\delta r}{\rho}\right\}\cdot f$: therefore $\frac{\frac{1}{f'} - \frac{1}{f}}{\delta r} = -\frac{1}{2\rho}\cdot\frac{1}{f}$; consequently $\frac{\int\frac{dm}{f'} - \int\frac{dm}{f}}{\delta r} = -\frac{1}{2\rho} \times \int\frac{dm}{f}$.

Since the spheroid is supposed to approach very nearly to the spherical figure, the radius of it will fall under this form of expression, viz. $\rho = a \times (1 + \alpha \cdot y)$; where a denotes the radius of a sphere concentric with the spheroid and nearly equal to it; α, a coefficient so small that its square and other higher powers may be neglected; and y, a function of two angles θ and ϖ which determine the position of ρ, θ being the angle contained by ρ and a fixt axis passing through the centre of the spheroid, and ϖ the angle which the plane drawn through ρ and the axis, makes with another plane passing by the same axis. Now, by substituting and neglecting all the terms of the order α^2 and the higher orders, the preceding values of V and $\left(\frac{dV}{dr}\right)$ will become

$$V = \frac{4\pi}{3}\cdot a^2\cdot(1 + 2\alpha\cdot y) + \int\frac{dm}{f}$$

$$\left(\frac{dV}{dr}\right) = -\frac{4\pi}{3}\cdot a\cdot(1 + \alpha\cdot y) - \frac{1}{2a}\cdot\int\frac{dm}{f}:$$

and, by combining these so as to exterminate $\int\frac{dm}{f}$, we shall get

$$\frac{1}{2}V + a\left(\frac{dV}{dr}\right) = -\frac{2\pi}{3}\cdot a^2$$

which is no other than Laplace's equation.*

* Liv. 3, No. 10. Equation (2).

We have here followed very closely all the steps of the demonstration contained in the *Mécanique Céleste*, and on first thoughts no reasoning can be more convincing, or appear more free from all obscurities. This much at least is certain, that every part of the demonstration is placed beyond the reach of all objections except the valuing of that term in the equation (B), which is derived from the difference between the spheroid and the sphere: and about this a deeper consideration of the nature of the functions concerned may raise in the mind some doubts and scruples. No better way can be devised for trying the soundness of LAPLACE's procedure, than to perform that part of the calculation which is alone liable to suspicion, without omitting any of the terms which he has tacitly rejected; to throw out such only as on examination can be proved to be necessarily evanescent when $\delta r = o$; and to retain the rest if there be any of a different description. Now, to apply this rule, we have $f^2 = 2\rho^2(1-\gamma)$; and $f'^2 = (\rho + \delta r)^2 - 2\rho(\rho + \delta r)\gamma + \rho^2 = \left\{1 + \frac{\delta r}{\rho}\right\} . 2\rho^2(1-\gamma) + \delta r^2$; therefore $f'^2 - \delta r^2 = \left\{1 + \frac{\delta r}{\rho}\right\} . f^2$: consequently, $\frac{1}{f} = \frac{1}{f'} \times \left\{1 + \frac{\delta r}{\rho}\right\}^{\frac{1}{2}} \times \left\{1 - \frac{\delta r^2}{f^2}\right\}^{-\frac{1}{2}}$; and, by expanding the second radical into a series, the complete value of $\frac{\frac{1}{f'} - \frac{1}{f}}{\delta r}$ will be equal to

$$\frac{1 - \left(1 + \frac{\delta r}{\rho}\right)^{\frac{1}{2}}}{\delta r} \times \frac{1}{f'} - \left(1 + \frac{\delta r}{\rho}\right)^{\frac{1}{2}} \times \left\{\frac{1}{2} . \frac{\delta r}{f'^3} + \frac{1.3}{2.4} . \frac{\delta r^3}{f'^5} + \&c.\right\};$$

and, by multiplying by dm and affixing the sign of integration, the complete value of $\frac{\int \frac{dm}{f'} - \int \frac{dm}{f}}{\delta r}$ will be equal to

$$\frac{1-\left(1+\frac{\delta r}{\rho}\right)^{\frac{1}{2}}}{\delta r}\cdot\int\frac{dm}{f'}-\left(1+\frac{\delta r}{\rho}\right)^{\frac{1}{2}}\cdot\left\{\frac{1}{2}\cdot\int\frac{\delta r\,.\,dm}{f'^{3}}+\frac{1.3}{2.4}\cdot\int\frac{\delta r^{3}.\,dm}{f'^{5}}+\text{\&c.}\right\}.$$

This expression being farther reduced into a series of simple terms, those terms will be included either in the form $\int\frac{\delta r^{i-1}.\,dm}{f'^{i+1}}$, or in the form $\delta r^{s}.\int\frac{\delta r^{i-1}.\,dm}{f'^{i+1}}$: whatever number i may denote, the first sort of terms, when they are integrated between the proper limits, will be found, on examination, to contain a part which, depending only on the nature of the molecules or of the function that expresses the thickness of the molecules, remains of the same magnitude for all values of δr; and consequently those terms do not necessarily vanish when $\delta r = o$: with respect to the second kind of terms, they are to be regarded as quantities of the same order with the multipliers written without the sign of integration, and they all vanish together with δr. Reserving till afterwards the proof of what has now been said, it is sufficient at present to have marked distinctly the characters of the quantities to be retained, and of those to be rejected. If then we retain the first sort of terms only and reject the rest, the value of $\frac{\int\frac{dm}{f'}-\int\frac{dm}{f}}{\delta r}$ will be equal to the series, viz.

$$-\frac{1}{2\rho}\cdot\int\frac{dm}{f'}-\frac{1}{2}\cdot\int\frac{\delta r\,.\,dm}{f'^{3}}-\frac{1.3}{2.4}\cdot\int\frac{\delta r^{3}.\,dm}{f'^{5}}-\text{\&c.}:$$

and, by substituting this in the equation (B), we shall get

$$\left(\frac{dV}{dr}\right)=-\frac{4\pi}{3}\cdot\rho-\frac{1}{2.\rho}\cdot\int\frac{dm}{f'}-\frac{1}{2}\cdot\int\frac{\delta r\,.\,dm}{f'^{3}}-\frac{1.3}{2.4}\cdot\int\frac{\delta r^{3}.\,dm}{f'^{5}}-\text{\&c.};$$

in which expression the value of all the terms under the sign of integration are to be taken on the supposition of $\delta r = o$. Finally if, in this last value of $\left(\frac{dV}{dr}\right)$ and the value of V already

found, we first substitute $a.(1+\alpha.y)$ for ρ, retaining only the quantities of the first order with regard to α; and then combine the two expressions so as to exterminate $\int\frac{dm}{f}$, we shall get the following equation instead of that of LAPLACE, viz.

$$\tfrac{1}{2}V+a\left(\frac{dV}{dr}\right)=-\frac{2\varpi}{3}.a^2-\tfrac{1}{2}a.\int\frac{\delta r.dm}{f^3}-\frac{1.3}{2.4}a.\int\frac{\delta r^3.dm}{f^5}-\&c. \text{(C)}.$$

2. In order to find the integrals in the equation (C), we must begin with seeking an analytical expression for the value of dm, which may be conceived to be a prism standing on an indefinitely small portion of the spherical surface, and limited in its height by the surface of the spheroid. Let ρ' denote the radius of the spheroid drawn to the molecule dm, and θ' and ϖ' the angles which determine the position of ρ' in like manner as θ and ϖ determine the position of ρ: and, if y' be put for the same function of θ' and ϖ' that y is of θ and ϖ, then $\rho'=a.(1+\alpha.y')$. Suppose θ' and ϖ', the arcs which determine the position of ρ', to vary; and the correspondent fluxion of the spherical surface whose radius is ρ, will be $=\rho^2.\sin.\theta'.d\theta'.d\varpi'=$ (μ' being put for cos. θ') $\rho^2.d\mu'.d\varpi'$; this is the base of the prism equal to dm: the height of the prism is plainly $=\rho'-\rho=\alpha.a.(y'-y)$: therefore $dm=\alpha.a.\rho^2.(y'-y).d\mu'.d\varpi'$: and, by substitution, the equation (C) will become

$$\tfrac{1}{2}V+a.\left(\frac{dV}{dr}\right)=-\frac{2\varpi}{3}.a^2-\alpha.a^2.\left\{\tfrac{1}{2}\iint\frac{\delta r.\rho^2.(y'-y).d\mu'.d\varpi'}{f'^3}\right.$$
$$\left.+\frac{1.3}{2.4}.\iint\frac{\delta r^3.\rho^2.(y'-y).d\mu'.d\varpi'}{f'^5}+\&c.\right\}\text{.....(D)}.$$

Since r, the distance of the attracted point from the centre, is $=\rho+\delta r$, and $f'=\left\{r^2-2r\rho.\gamma+\rho^2\right\}^{\frac{1}{2}}$; therefore the general term of the series in the last equation will be

$$\iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot y' \cdot d\mu' \cdot d\varpi'}{f'^{i+1}} - \iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot y \cdot d\mu' \cdot d\varpi'}{f'^{i+1}}:$$

and because y' is a function of the variable angles θ' and ϖ', or of μ' and ϖ'; and y is a constant quantity; therefore, if v' be put to denote a function of the angles θ' and ϖ', both the integrals in the general term will be obtained by investigating the integral

$$\iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot v' \cdot d\mu' \cdot d\varpi'}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{i+1}{2}}}$$

for the whole surface of the sphere, and in the particular circumstance of $r = \rho$, or $r - \rho = o$.

3. The formula which is now to be considered cannot be integrated without limiting the symbol v' to denote a particular function, or class of functions. But Laplace's demonstration will be completely overturned, if it shall be shown that, in any hypothesis for v', the formula in question has a finite value when $r - \rho = o$: for then the only reason which he can be supposed to assign for rejecting such terms in the value of $\left(\frac{dV}{dr}\right)$; namely, that they contain a vanishing factor, must be allowed to be inconclusive. We shall henceforth suppose that v' denotes a rational and integral function of μ', $\sqrt{1-\mu'^2}$. cos. ϖ', $\sqrt{1-\mu'^2}$. sin. ϖ', which are three rectangular co-ordinates of a point in the surface of a sphere; a supposition which in effect embraces the whole extent of Laplace's method.

The demonstration which Laplace has given of his fundamental theorem is independent on the function y, being drawn entirely from the nature of the algebraic expression of the distance between the attracted point and a molecule of the

matter spread over the surface of the sphere.* From this circumstance indeed is derived one great advantage of his method, namely its great generality; for no restriction whatever is imposed on the nature of the spheroid excepting that of a near approach to the spherical figure. Nevertheless the author, by means of a simple transformation, immediately deduces from his theorem an equation which proves that y and V are expressed by two series both containing the same sort of terms:† and since all the terms of the series for V can only be rational and integral functions of μ, $\sqrt{1-\mu^2}$. cos. ϖ, $\sqrt{1-\mu^2}$. sin. ϖ;‡ it follows that y must be a like function of the same three quantities. We may remark here that this consequence of LAPLACE's reasoning appears to be inconsistent with the premises: for it is hard to reconcile with the rules of legitimate deduction that an equation obtained by supposing y to be arbitrary, should, merely by having its form changed, be made to prove that the same quantity must be restricted to signify a function of a particular kind. But we mention this only by the bye, without meaning to insist upon it; although we cannot help thinking that it ought to have led the learned author to entertain suspicions of the accuracy of his calculations; all that we intend by the foregoing observation is to prove that in point of fact we shall embrace the whole extent of LAPLACE's method by supposing y to be a rational and integral function of three rectangular co-ordinates of a point in the surface of a sphere.

Supposing then υ' to denote such a function as has been mentioned, we are to investigate the value of this integral,

* Liv. 3e, No. 10. † Liv. 3e, No. 11. ‡ Liv. 3e, No. 9.

viz.

$$\iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot v' \cdot d\mu' \cdot d\varpi'}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{i+1}{2}}}$$

when it is extended to the whole surface of the sphere, and in the particular circumstance of $r = \rho$, or $r - \rho = o$. We must begin with transforming the formula to be integrated. The arcs θ and θ' are the two sides of a triangle formed on the surface of a sphere; the angle contained by those sides is $\varpi' - \varpi$; and the third side of the same triangle is no other than the arc whose cosine has been denoted by γ: let ϕ denote the angle opposite to the side θ' whose cosine is μ'; then if we suppose θ' and ϖ' to vary, it has already been proved that the correspondent fluxion of the surface of the sphere will be $= \rho^2 \cdot d\mu' \cdot d\varpi'$; but if we make γ and ϕ vary, the same fluxion will be $= \rho^2 \cdot d\gamma \cdot d\phi$: therefore

$$\frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot v' \cdot d\mu' \cdot d\varpi'}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{i+1}{2}}} = \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot v' \cdot d\gamma \cdot d\phi}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{i+1}{2}}}:$$

and as this is true for every element of the spherical surface, the fluents will likewise be equal when they are extended to the whole surface of the sphere. To complete the transformation we must next convert v' into a function of γ and ϕ; after which the integration with regard to ϕ will be independent of the denominator in which γ only is contained. Suppose v' to be actually transformed as here mentioned, then

$$\iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot d\mu' \cdot d\varpi'}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{i+1}{2}}} = \int \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot d\gamma \cdot \int v' \cdot d\phi}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{i+1}{2}}};$$

the sign of integration in the numerator being understood to affect the variable ϕ only.

For the greater simplicity we shall first consider the case when υ' is a rational and integral function of μ' only without ϖ', as is the case in spheroids of revolution. Suppose then $\upsilon' = \mathrm{F}(\mu')$: and by spherical trigonometry,

$$\mu' = \mu\gamma + \sqrt{1-\mu^2}\,.\,\sqrt{1-\gamma^2}\,.\,\cos.\,\phi;$$

therefore by TAYLOR's theorem,

$$\upsilon' = \mathrm{F}(\mu\gamma) + (1-\mu^2)^{\frac{1}{2}}(1-\gamma^2)^{\frac{1}{2}}\,.\,\frac{d\,.\,\mathrm{F}(\mu\gamma)}{d(\mu\gamma)}\,.\,\cos.\,\phi$$
$$+ (1-\mu^2)^{\frac{2}{2}}(1-\gamma^2)^{\frac{2}{2}}\,.\,\frac{d^2\,.\,\mathrm{F}(\mu\gamma)}{d(\mu\gamma)^2}\,.\cos.^2\phi + \&c.^*$$

and by substituting for the powers of cos. ϕ their values in the multiple arcs, we shall have,

$$\upsilon' = \Gamma^{(0)} + (1-\mu^2)^{\frac{1}{2}}\,.\,(1-\gamma^2)^{\frac{1}{2}}\,.\,\Gamma^{(1)}\,.\,\cos.\,\phi$$
$$+ (1-\mu^2)^{\frac{2}{2}}\,.\,(1-\gamma^2)^{\frac{2}{2}}\,.\,\Gamma^{(2)}\,.\,\cos.\,2\phi + \&c.$$

the general term being $(1-\mu^2)^{\frac{i}{2}}\,.\,(1-\gamma^2)^{\frac{i}{2}} \times \Gamma^{(i)}\,.\,\cos.\,i\phi$, where $\Gamma^{(i)}$ represents a rational and integral function of γ. Now if we multiply by $d\phi$, and then integrate between the limits $\phi = o$ and $\phi = 2\pi$, we shall get $\int \upsilon' d\phi = 2\pi\,.\,\Gamma^{(0)}$; because the integrals of all the terms which contain the cosines of the multiple arcs are evanescent at both the limits. Therefore, by substitution,

$$\iint \frac{(r-\rho)^{i-1}\,.\,\rho^2\,.\,\upsilon'\,.\,d\mu'\,.\,d\varpi'}{\left\{r^2 - 2r\rho\,.\,\gamma + \rho^2\right\}^{\frac{i+1}{2}}} = 2\pi\,.\int \frac{(r-\rho)^{i-1}\,.\,\rho^2\,.\,\Gamma^{(0)}\,.\,d\gamma}{\left\{r^2 - 2r\rho\,.\,\gamma + \rho^2\right\}^{\frac{i+1}{2}}}.$$

In order to execute the remaining integration I remark that $f = \left\{r^2 - 2r\rho\,.\,\gamma + \rho^2\right\}^{\frac{1}{2}}$, and $d\gamma = -\frac{fdf}{r\rho}$: therefore by conti-

* By the notation $\frac{d^n\,.\,\mathrm{F}(\mu\gamma)}{d(\mu\gamma)^n}$ it is to be understood that in taking the fluxions, $(\mu\gamma)$ is to be considered as one simple quantity; the same as if it were represented by a single letter.

nually exterminating $d\gamma$, and integrating partially with regard to f, we shall obtain

$$\iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot v' \cdot d\mu' \cdot d\varpi'}{\left\{r^2-2r\rho \cdot \gamma+\rho^2\right\}^{\frac{i+1}{2}}} = 2\pi \cdot \left\{\frac{\rho}{r\rho} \cdot \frac{(r-\rho)^{i-1}}{f^{i-1}} \cdot \frac{\Gamma^{(0)}}{i-1} - \frac{\rho^2}{r^2\rho^2} \cdot \right.$$

$$\left. \frac{(r-\rho)^{i-1}}{f^{i-3}} \cdot \frac{\frac{d\Gamma^{(0)}}{d\gamma}}{i-1 \cdot i-3} + \frac{\rho^3}{r^3\rho^3} \cdot \frac{(r-\rho)^{i-1}}{f^{i-5}} \cdot \frac{\frac{d^2\Gamma^{(0)}}{d\gamma^2}}{i-1 \cdot i-3 \cdot i-5} - \text{\&c.} \right\}.$$

This fluent, which, it is to be observed, increases as γ increases, is to be taken between the limits $\gamma = -1$ and $\gamma = 1$: at the first limit $\gamma = -1$, every term of the fluent is evanescent when $r-\rho = 0$: at the second limit, $\gamma = 1$ and $f = r-\rho$, every term is likewise evanescent except the first, which is $2\pi \times \frac{(r-\rho)^{i-1}}{(r-\rho)^{i-1}} \times \frac{\Gamma^{(0)}}{i-1} = 2\pi \cdot \frac{\Gamma^{(0)}}{i-1}$, for all values of r, and even when $r-\rho = 0$: therefore

$$\iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot v' \cdot d\mu' \cdot d\varpi'}{\left\{r^2-2r\rho \cdot \gamma+\rho^2\right\}^{\frac{i+1}{2}}} = \frac{2\pi}{i-1} \cdot \Gamma^{(0)}:$$

observing that we must make $\gamma = 1$ in the function $\Gamma^{(0)}$. Now the suppositions $\gamma = -1$ and $\gamma = 1$, correspond to $\mu' = -\mu$ and $\mu' = \mu$: and therefore if we put v to denote the same function of μ that v' does of μ'; that is, if v represent what v' becomes when $\mu' = \mu$; then it will follow, from the nature of the transformed value of v', that $v = \Gamma^{(0)}$ when $\gamma = 1$, because all the other terms are equal to nothing for this value of γ: therefore finally

$$\iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot v' \cdot d\mu' \cdot d\varpi'}{\left\{r^2-2r\rho \cdot \gamma+\rho^2\right\}^{\frac{i+1}{2}}} = \frac{2\pi}{i-1} \cdot v.$$

We shall now pass on to the general case when v' is a

rational and integral function of μ', $\sqrt{1-\mu'^2}\,.\cos.\varpi'$, $\sqrt{1-\mu'^2}\,.\sin.\varpi'$. Let x, y, z stand for μ', $\sqrt{1-\mu'^2}\,.\cos.\varpi'$, $\sqrt{1-\mu'^2}\,.\sin.\varpi'$; and x', y', z' for the analogous magnitudes γ, $\sqrt{1-\gamma^2}\,.\cos.\phi$, $\sqrt{1-\gamma^2}\,.\sin.\phi$: the first set of quantities are three rectangular co-ordinates of a point in the surface of the sphere whose radius is unit, drawn to the planes of three great circles two of which intersect in the origin of the arcs whose cosines are μ' and μ; and the second set are the three rectangular co-ordinates of the same point as before referred to three other planes two of which pass through the origin of the arc whose cosine is γ: therefore, in order to obtain the relation of these two sets of quantities we have only to apply the method for *transforming the co-ordinates*: in this manner we shall readily obtain,

$$x = x'.\mu + y'.\sqrt{1-\mu^2}$$
$$y = x'.\sqrt{1-\mu^2}\,.\cos.\varpi - y'.\mu.\cos.\varpi - z'.\sin.\varpi$$
$$z = x'.\sqrt{1-\mu^2}\,.\sin.\varpi - y'.\mu.\sin.\varpi + z'.\cos.\varpi.$$

Because v' is a rational and integral function of x, y, z; by substituting the values of these quantities just investigated, it will be converted into a like function of x', y', z', that is, of γ, $\sqrt{1-\gamma^2}\,.\cos.\phi$, $\sqrt{1-\gamma^2}\,.\sin.\phi$: and farther, if the several powers and products of cos. ϕ and sin. ϕ be exterminated by means of the equivalent expressions in the sines and cosines of the multiple arcs, the expression v', after all the terms are properly arranged, will assume the following form, viz.

$$v' = \Gamma^{(0)} + (1-\mu^2)^{\frac{1}{2}}.(1-\gamma^2)^{\frac{1}{2}}.\Gamma^{(1)}.\cos.\phi + (1-\mu^2)^{\frac{2}{2}}.(1-\gamma^2)^{\frac{2}{2}}.\Gamma^{(2)}.\cos.2\phi + \&c.$$
$$+ (1-\mu^2)^{\frac{1}{2}}.(1-\gamma^2)^{\frac{1}{2}}.\Delta^{(1)}.\sin.\phi + (1-\mu^2)^{\frac{2}{2}}.(1-\gamma^2)^{\frac{2}{2}}\,\Delta^{(2)}.\sin.2\phi + \&c.$$

the general term being

$(1-\mu^2)^{\frac{i}{2}} . (1-\gamma^2)^{\frac{i}{2}} . \Gamma^{(i)} . \cos. i\varphi + (1-\mu^2)^{i} . (1-\gamma^2)^{\frac{i}{2}} . \Delta^{(i)}$
$. \sin. i\varphi,$

where $\Gamma^{(i)}$ and $\Delta^{(i)}$ represent rational and integral functions of γ. Now if we multiply by $d\varphi$ and then integrate from $\varphi = o$ to $\varphi = 2\pi$, we shall obtain as before $\int v' d\varphi = 2\pi \times \Gamma^{(o)}$; because the integrals of all the terms multiplied by the cosines and sines of the multiple arcs are of the same magnitude at both the limits. Therefore, by following exactly the same procedure as before, we shall arrive at this equation, viz.

$$\iint \frac{(r-\rho)^{i-1} . \rho^2 . v' . d\mu' . d\varpi'}{\left\{r^2 - 2r\rho . \gamma + \rho^2\right\}^{\frac{i+1}{2}}} = \frac{2\pi}{i-1} . \Gamma^{(o)}$$

in which the function $\Gamma^{(o)}$ is to be valued on the supposition that $\gamma = 1$. But the suppositions $\gamma = -1$, $\varphi = o$, correspond to $\mu' = -\mu$, $\varpi' = \varpi$; and the suppositions $\gamma = 1$, $\varphi = 2\pi$, correspond to $\mu' = \mu$, and $\varpi' = \varpi + 2\pi$: therefore if v denote what v' becomes when $\mu' = \mu$ and $\varpi' = \varpi + 2\pi$: that is if v be the same function of μ, $\sqrt{1-\mu^2} . \cos. \varpi$, $\sqrt{1-\mu^2} . \sin. \varpi$ that v' is of μ', $\sqrt{1-\mu'^2} . \cos. \varpi'$, $\sqrt{1-\mu'^2} . \sin. \varpi'$; it is plain, from the transformed value of v', that $v = \Gamma^{(o)}$ when $\gamma = 1$. Therefore, we shall have

$$\iint \frac{(r-\rho)^{i-1} . \rho^2 . v' . d\mu' . d\varpi'}{\left\{r^2 - 2r\rho . \gamma + \rho^2\right\}^{\frac{i+1}{2}}} = \frac{2\pi}{i-1} . v. \quad \text{(E)}.$$

4. The investigation just gone through shows how necessary it is to retain all the terms we have done in the equation (C), and at the same time it proves that the terms thrown out in finding that equation were justly rejected. It completely

overturns the demonstration of LAPLACE; since in his procedure an infinite number of terms are neglected merely because they are multiplied by some power of the evanescent quantity δr; a reason which the preceding analysis demonstrates in the clearest manner to be altogether inconclusive.

Nevertheless, if we now suppose that y' is a rational and integral function of μ', $\sqrt{1-\mu'^2} \cdot \cos. \varpi'$, $\sqrt{1-\mu'^2} \cdot \sin. \varpi'$, and, by the help of the formula (E), inquire into the values of the several terms in the series on the right-hand side of the equation (D), we shall find that LAPLACE's equation is rigorously true in that hypothesis. For, as we have already shewn the general term of the series consists of these two integrals, viz.

$$\iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot y' \cdot d\mu' \cdot d\varpi'}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{i+1}{2}}} - \iint \frac{(r-\rho)^{i-1} \cdot \rho^2 \cdot y \cdot d\mu' \cdot d\varpi'}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{i+1}{2}}}$$

which being valued separately, the result will be,

$$\frac{2\pi}{i-1} \cdot y - \frac{2\pi}{i-1} \cdot y = 0:$$

therefore the right-hand side of the equation (D) will be reduced to its first term, and we shall have

$$\tfrac{1}{2} \mathrm{V} + a \cdot \left(\frac{d\mathrm{V}}{dr}\right) = -\frac{2\pi}{3} \cdot a^2 *$$

the very equation of LAPLACE.

But although the proposition in the *Mécanique Céleste* is thus found to be true in one particular hypothesis, the arguments, that have been urged against the proof of it contained in that work, lose none of their force. It appears indeed that the quantities which LAPLACE has omitted are really equal to nothing in one kind of spheroids; yet this does not happen for any reason which he has assigned, but for a reason which has

* Liv. 3e, No. 10. Equat. (2).

no manner of connection with any thing touched upon in the whole course of his demonstration. In the rigorous investigation, the rules of the integral calculus are necessary; whereas the reasoning of LAPLACE requires only the direct method of fluxions. Besides his proof goes too far; for it applies to all spheroids that approach nearly to the spherical figure: but the method, when it is strictly analyzed, is limited to those spheroids of the same description which have their radii expressed by rational and integral functions of three rectangular co-ordinates of a point in the surface of a sphere. We may even infer from what LAPLACE himself has proved that his method is confined exclusively to such spheroids: for he has shewn that the expression for y is not arbitrary, but that it depends upon the series for V;* whence it follows that it can only be such a function as is mentioned above, and as we have supposed it to be.

5. In order still farther to confirm the conclusions already obtained I shall now show that LAPLACE's method for the attractions of spheroids that differ but little from spheres is contained in the formula (E) from which it may be deduced without the intervention of his theorem relating to the attraction at the surface.

Conceive a spheroid whose radius is $\rho = a \cdot (1 + \alpha \cdot y)$ as before; and also a sphere, whose radius is a, concentric with the spheroid; and let r denote the distance of an attracted point situate in the prolongation of ρ, from the common centre: then the value of V relatively to the sphere will be $= \frac{4\pi}{3} \cdot \frac{a^3}{r}$: and if dm denote one of the molecules of the excess of the

* Liv. 3, No. 11.

spheroid above the sphere; the value of the same function, relatively to that excess will be $= \int \frac{dm}{\left\{r^2 - 2ra \cdot \gamma + a^2\right\}^{\frac{1}{2}}} = \int \frac{dm}{f}$: therefore,

$$V = \frac{4\pi}{3} \cdot \frac{a^3}{r} + \int \frac{dm}{f}.$$

Let $\rho' = a \cdot (1 + \alpha \cdot y')$ be the radius of the spheroid drawn to the molecule dm; then the thickness of the molecule will be $= \alpha \cdot a \cdot y'$, and $dm = \alpha \cdot a^3 \cdot y' \cdot d\mu' \cdot d\varpi'$: again, if we expand $\frac{1}{f}$ into a series of terms containing the descending powers of r, as LAPLACE has done,* we shall have

$$\frac{1}{f} = \frac{1}{r} \cdot Q^{(0)} + \frac{a}{r^2} Q^{(1)} + \frac{a^2}{r^3} \cdot Q^{(2)} + \&c.$$

$Q^{(i)}$ denoting generally such a function of μ and ϖ as satisfies his equation in partial fluxions: and if we farther put $\int Q^{(i)} \cdot dm = \alpha \cdot a^3 \cdot \iint Q^{(i)} \cdot y' \cdot d\mu' \cdot d\varpi' = \alpha \cdot a^3 \cdot U^{(i)}$, we shall get

$$V = \frac{4\pi}{3} \cdot \frac{a^3}{r} + \alpha \cdot \frac{a^3}{r} \cdot \left\{U^{(0)} + \frac{a}{r} \cdot U^{(1)} + \frac{a^2}{r^2} \cdot U^{(2)} + \&c.\right\}:$$

and $U^{(i)}$ will satisfy the same equation in partial fluxions that $Q^{(i)}$ does.

Moreover suppose r to vary and equate the fluxions of $\frac{1}{f}$ and of the series equal to it; and after having multiplied by r, the result will be as follows:

$$\frac{r^2 - ra \cdot \gamma}{f^3} = \frac{1}{r} \cdot Q^{(0)} + \frac{a}{r^2} \cdot 2Q^{(1)} + \frac{a^2}{r^3} \cdot 3Q^{(2)} + \&c.:$$

but $-ra \cdot \gamma = \frac{1}{2} \cdot f^2 - \frac{1}{2} r^2 - \frac{1}{2} a^2$; therefore, by substitution we shall readily get

$$\frac{r^2 - a^2}{2f^3} = -\frac{1}{2} \cdot \frac{1}{f} + \frac{1}{r} \cdot Q^{(0)} + \frac{a}{r^2} \cdot 2Q^{(1)} + \frac{a^2}{r^3} \cdot 3Q^{(2)} + \&c.;$$

* Liv. 3e, No. 9.

and if we first substitute for $\frac{1}{f}$ the series equal to it; next multiply by $a.y'.d\mu'.d\varpi'$; and then integrate; we shall finally obtain

$$\frac{r+a}{2a}.\iint\frac{(r-a)\ a^2.y'.d\mu'.d\varpi'}{\left\{r^2-2ra.\gamma+a^2\right\}^{\frac{3}{2}}}=\frac{1}{2}\left\{\frac{a}{r}.U^{(0)}+\frac{a^2}{r^2}.3U^{(1)}+\frac{a^3}{r^3}.5U^{(2)}+\&c.\right\}$$

now by the formula (E) the value of the integral on the left-hand side, when $r=a$, is $=2\pi y$: therefore

$$4\pi y=U^{(0)}+3U^{(1)}+5U^{(2)}+7U^{(3)}+\&c.$$

a formula which is equivalent to what LAPLACE has deduced in his manner,* and which is the foundation of his very ingenious method. In effect, if we develope the given function y, as LAPLACE has taught us to do,† into a series of terms every one of which shall satisfy his equation in partial fluxions; so that

$$y=Y^{(0)}+Y^{(1)}+Y^{(2)}+\&c.;$$

then, since it is proved that this expansion is unique, by equating the like terms of the two values of y, we shall have generally,

$$4\pi.Y^{(i)}=(2i+1).U^{(i)};$$

by means of which all the quantities $U^{(0)}$, $U^{(1)}$, $U^{(2)}$ &c. which are the coefficients in the series for V, will become known.

This analysis proves in the clearest manner that LAPLACE's method is exact only in one hypothesis for y, and that it is strictly confined to one class of spheroids: for it can hardly be maintained that the formula (E) will be true whatever function the symbol v' may be supposed to denote.

* Liv. 3e, No. 11. † Liv. 3e, No. 16.

6. We have hitherto confined our attention to the law of attraction that actually takes place in nature; but before we conclude this discourse it may not be improper to add a few words on the theorem taken in the general sense in which it is laid down in the *Mécanique Céleste.** Let n represent the exponent of that power of the distance according to which the attraction acts; dM a molecule of the spheroid, and f the distance of the molecule from the attracted point; then $V = \int . f^{n+1} . dM$, the fluent being extended to all the molecules in the mass of the spheroid. If ρ denote a radius of the spheroid and r the distance of an attracted point (situate in the prolongation of ρ) from the centre, the function V will consist of two parts one derived from the sphere whose radius is ρ; and the other, which we shall denote separately by s, from the difference between the spheroid and the sphere: and if dm denote one of the molecules of that difference, then $s = \int . f^{n+1} . dm$: therefore $\left(\frac{ds}{dr}\right) = (n+1) \int \frac{\left(\frac{df}{dr}\right)}{f} . f^{n+1} . dm$: but retaining the same denominations as before, $f = \left\{r^2 - 2r\rho . \gamma + \rho^2\right\}^{\frac{1}{2}}$; and $\frac{\left(\frac{df}{dr}\right)}{f} = \frac{r - \rho . \gamma}{r^2 - 2r\rho . \gamma + \rho^2}$: therefore

$$\left(\frac{ds}{dr}\right) = (n+1) . \int \frac{r - \rho . \gamma}{r^2 - 2r\rho . \gamma + \rho^2} . f^{n+1} . dm:$$

and, by substituting $\rho\ (1 - \gamma) + (r - \rho)$ for $r - \rho . \gamma$, we shall get

$$\left(\frac{ds}{dr}\right) = (n+1) . \int \frac{\rho(1-\gamma)}{r^2 - 2r\rho . \gamma + \rho^2} . f^{n+1} . dm + (n+1) . (r - \rho) . \int . f^{n-1} . dm:$$

* Liv. 3e, No. 10. Equation (1).

when the attracted point is in the surface then $r = \rho$, and the preceding expressions for s and $\left(\frac{ds}{dr}\right)$ will become

$$s = \int . \left\{2\rho^2 (1-\gamma)\right\}^{\frac{n+1}{2}} . dm$$

$$\left(\frac{ds}{dr}\right) = \frac{n+1}{2\rho} . \int . \left\{2\rho^2 (1-\gamma)\right\}^{\frac{n+1}{2}} . dm + (n+1) . (r-\rho) . \int . f^{n-1} . dm$$

observing that the second term on the right-hand side of the latter formula is to be valued on the supposition of $r - \rho = o$: therefore, by combining the two formulas, we shall get

$$\left(\frac{ds}{dr}\right) - \frac{n+1}{2\rho} . s = (n+1) . (r-\rho) . \int . f^{n-1} . dm. \quad (\text{F})$$

When n is equal to unit or greater than unit, it is plain that the quantity under the sign of integration in equation (F) will have a finite value at both the limits; and therefore, on account of the vanishing factor $(r-\rho)$, that side of the equation will be equal to nothing. Consequently by putting a for ρ, which is permitted (because s is of the order α), we shall get $\left(\frac{ds}{dr}\right) - \frac{n+1}{2a} . s = o$: whence it follows that the value of the function $\left(\frac{dV}{dr}\right) - \frac{n+1}{2a} . V$ will depend only upon the sphere whose radius is ρ; since the part of that function which is derived from the difference between the spheroid and sphere has been proved to be equal to nothing; which is in other words the theorem of LAPLACE.

The demonstration we have just gone through is drawn from the same considerations as that contained in the *Mécanique Céleste*, from which it does not differ so much in spirit as in the manner of stating the reasoning. It must therefore be admitted that, when the exponent of the law of attraction is

positive and not less than unit, the proof of LAPLACE is not liable to much objection; and that his theorem is true to the full extent of the enunciation, or for all spheroids that differ but little from spheres, whatever be the function which expresses the thickness of the molecules in the excess of the spheroid above the sphere.

Let us next examine what will happen when the exponent of the law of attraction is negative: for this purpose, write $-n$ for n in the equation (F), and it will become

$$\left(\frac{ds}{dr}\right) + \frac{n-1}{2a} \cdot s = -\frac{n-1}{(r-\rho)^{n-2}} \cdot \int \frac{(r-\rho)^{n-1} \cdot dm}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{n+1}{2}}} :$$

now, according to what has already been proved, the expression under the sign of integration must be regarded as a finite quantity depending on the nature of the molecule dm: therefore, when n is greater than 2, the part of the function $\left(\frac{dV}{dr}\right) + \frac{n-1}{2a} \cdot V$, which is derived from the difference between the spheroid and sphere, will, on account of the infinite factor, be infinitely great instead of being equal to nothing, as LAPLACE's theorem would require it to be.

The case of nature corresponds to the supposition of $n = 2$ in the last formula; in this case, after having multiplied by a, we shall find

$$a\left(\frac{ds}{dr}\right) + \tfrac{1}{2}s = -a \cdot \int \frac{(r-\rho) \cdot dm}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{1}{2}}} :$$

whence we get the value of that part of the function $\frac{1}{2}V + a \cdot \left(\frac{dV}{dr}\right)$, which is derived from the difference between the spheroid and the sphere: but the value of the other part, which is derived from the sphere, is $= -\frac{2\pi}{3} \cdot a^2$: consequently

$$\tfrac{1}{2}V + a \cdot \left(\frac{dV}{dr}\right) = -\frac{2\pi}{3} \cdot a^2 - a \cdot \int \frac{(r-\rho) \cdot dm}{\left\{r^2 - 2r\rho \cdot \gamma + \rho^2\right\}^{\frac{1}{2}}}:$$

In this formula the expression under the sign of integration is a finite quantity* depending on the nature of the molecules; and thus the case of nature is the point where the reasoning of LAPLACE ceases to be exact.

The equation last investigated, although it has a finite form ought nevertheless to be equivalent to the equation (C) which is expressed in an infinite series. To bring this matter to the proof, I observe that both the equations will be accurate whether ρ reaches exactly to the surface of the spheroid, or only nearly to that surface: for all that the reasoning supposes is that ρ differs only by the small quantity $\alpha \cdot a \cdot y$ from a; that the attracted point is in the surface of the sphere of which ρ is the radius; and that the shell of matter spread over the surface of the same sphere is every where so thin as to contain only one molecule in the depth. Suppose then v' to denote a rational and integral function of μ', $\sqrt{1-\mu'^2} \cdot \cos. \varpi'$, $\sqrt{1-\mu'^2} \cdot \sin. \varpi'$; and let $\alpha \cdot a \cdot v'$ denote the thickness of the molecule dm; then $dm = \alpha \cdot a \cdot \rho^2 \cdot v' \cdot d\mu' \cdot d\varpi'$; consequently, on account of the formula (E), the equation last found will become

$$\tfrac{1}{2}V + a \cdot \left(\frac{dV}{dr}\right) = -\frac{2\pi}{3} \cdot a^2 - \alpha \cdot a^2 \cdot 2\pi \cdot v:$$

and in like manner by valuing the several terms of the equation (C) we shall get

$$\tfrac{1}{2}V + a \cdot \left(\frac{dV}{dr}\right) = -\frac{2\pi}{3} \cdot a^2 - \alpha \cdot a^2 \cdot 2\pi \cdot \left\{\tfrac{1}{2}v + \frac{1.3}{2.4} \cdot \frac{v}{3} + \frac{1.3.5}{2.4.6} \cdot \frac{v}{5} + \&c.\right\}:$$

* Art. 3. Equat. (E).

now the exact coincidence of these two equations is proved by observing that the series into which v is multiplied is equal to unit; for it is equal to $1 - \sqrt{1-x}$ when x is put equal to unit.

7. I have now explained at sufficient length my objections to LAPLACE's demonstration, and the reasons on which they are founded. The subject is abstruse and subtile; on which account I have taken all the pains I could to make the processes as clear as the nature of such a discussion would permit; and I have endeavoured to confirm the conclusions I wished to establish by investigating them in more ways than one. It appears, from what has been shown, that LAPLACE's theorem, which in the law of attraction that takes place in nature is contained in Equation (2), No. 10, Liv. 3e. of the *Mécanique Céleste*, is neither true of all spheroids that nearly approach the spherical figure as the author thought, nor is it strictly demonstrated in any case. It is exclusively confined to that class of spheroids which, while they differ little from spheres, likewise have their radii expressed by rational and integral functions of a point in the surface of a sphere: in this hypothesis LAPLACE's equation has been rigorously demonstrated in the preceding pages; and it is to such spheroids only that his ingenious method, which is founded on that equation, can be applied.

And here a question occurs. Since the solution of the problem of attractions contained in the *Mécanique Céleste* is not a universal method for all spheroids differing little from spheres, as the author conceived it to be, but is really limited to one particular class of spheroids; it may be asked, how far will this limitation affect the physical theories he has built on his

method? On this question I shall confine myself to the two following remarks.

In the first place the method we are speaking of is entirely unfit for finding *a priori* by a direct analysis all the possible figures compatible with a state of permanent equilibrium: for it is exclusively confined to spheroids whose radii are rational and integral functions of three rectangular co-ordinates of a point in the surface of a sphere, and it can only be employed to detect such figures belonging to that class as will satisfy the required conditions. On this account the analysis in No. 25, Liv. 3e, cannot be admitted as satisfactory: and indeed from the words in the beginning of No. 26, we may infer that the author himself was not perfectly satisfied with the strictness and universality of his investigation.

But, in the second place, although it cannot be granted that the method of Laplace is general for all spheroids that nearly approach the spherical figure, it is nevertheless very extensive, and is applicable to a great variety of cases comprehending figures of revolution as well as others to which that character does not belong. In the class of spheroids that falls within the scope of the method, the algebraic expression of the radius may contain an indefinite number of terms and arbitrary coefficients; on which account that class may be considered as embracing within its limits all round figures that differ little from spheres, if not exactly, at least as nearly as may be required. In this point of view therefore the real utility and value of Laplace's solution of the problem of attractions will not be much diminished by its failing in that degree of generality which its author conceived it to possess.

In concluding this discourse, I have only farther to recom-

mend the following observations to the notice of such mathematicians as may devote some part of their attention to the cultivation of this important branch of physics. Although the analysis which LAPLACE has traced out for the attractions of spheroids must be allowed to be very ingenious and masterly, yet still there are some considerations which cannot but lead us to think, that it falls short of that degree of perfection which it is laudable to aim at. And in particular the coefficients of the several terms of the expansion are, in his procedure, formed one after another, beginning with the last term: so that the first terms of the series cannot be found without previously computing all the rest. This is no doubt an imperfection of some moment: and it can only be removed by deducing every term of the series immediately from the radius of the spheroid, and enabling the analyst to calculate any proposed coefficient independently of all the rest by a process, as easy at least as in the investigation of LAPLACE. It is also to be observed, that in the application of this method we are not limited to such spheroids as do not differ much from spheres; we may extend it to all spheroids provided their radii be expressed by functions of the kind so often mentioned: it would therefore be extremely desirable to deduce in this way all the known formulas for ellipsoids, and elliptical spheroids of revolution, which would bring the whole theory of attractions under one uniform analysis.

APPENDIX

TO THE PRECEDING PAPER.

Read Nov. 7, 1811.

SOME time before the end of May last, a paper of mine was presented to the Royal Society, in which I entered on an examination of a fundamental proposition in the second chapter of the third book of the *Mécanique Céleste.* About three months after that paper was in the possession of the Society, towards the middle of August, a large collection of foreign books, imported from the Continent, was received in London; among which there were several *Cahiers* of the *Journal de l'Ecole Polytechnique.* In the 15th *Cahier,* which had been published at Paris in December 1809, although it did not find its way into this country prior to the above date, there is a short memoir by LAGRANGE on the same subject treated of in my paper: and in this Appendix I shall lay before the Society a short account of LAGRANGE's memoir, pointing out what are the views of that celebrated mathematician in regard to the conclusions obtained in my paper.

LAGRANGE prefixes this title to his memoir, viz. "Eclair- "cissement d'une difficulté singulière qui se rencontre dans "le calcul de l'Attraction des Spheroides très peu différens de "la Sphère." In order to avoid the explaining of new notations I shall make use of the symbols employed in my paper.

1. If we put $s = \int \frac{dm}{f} = \alpha . a^3 . \iint \frac{v' . d\mu' . d\varpi'}{\sqrt{r^2 - 2ra . \gamma + a^2}}$; the equa-

tion $V = \frac{4\pi a^3}{3r} + \int \frac{dm}{f}$, obtained in No. 5 of the preceding paper will become

$$V = \frac{4\pi a^3}{3r} + s:$$

which is equivalent to the equation in the first paragraph of No. 2 of LAGRANGE's memoir, the only difference being in the characters employed. And if we treat this equation as in No. 6 of my paper, and suppose that the term multiplied by $r - a$ vanishes when $r = a$, we shall get

$$\tfrac{1}{2} V + a \left(\frac{dV}{dr}\right) = - \frac{2\pi . a^2}{3}$$

$$\tfrac{1}{2} s + a \left(\frac{ds}{dr}\right) = o$$

when the attracted point is in the surface of the spheroid: and these equations are the very same with those which LAGRANGE has investigated, by a process entirely similar, in the remaining part of No. 2.

The equation $\frac{1}{2} s + a \left(\frac{ds}{dr}\right) = o$, is considered by LAGRANGE in No. 3. As this equation was obtained by reasonings which are independent on the nature of the molecules in the difference between the spheroid and the sphere, it ought to be true for all values of the function v' which expresses the thickness of those molecules. In order to examine this point, LAGRANGE supposes v' to be a constant quantity; and on this supposition he finds that in fact the equation $\frac{1}{2} s + a \left(\frac{ds}{dr}\right) = o$, does not take place, but that the true equation is $\frac{1}{2} s + a \left(\frac{ds}{dr}\right) = - 2\pi . a^2 . v$. Here then there is certainly a great difficulty: for the very same reasonings which prove $\frac{1}{2} s + a \left(\frac{ds}{dr}\right) = o$, on the supposition of LAPLACE that the sphere touches the spheroid at

the attracted point, will likewise prove that the same equation is true, when the solids do not touch, and when v' is constant, or has any other value whatever.

At present I shall pass by what is said in Nos. 4 and 5 of LAGRANGE's memoir, on which I shall offer some remarks below. In No. 6 he proceeds to inquire into the reason of the difficulty or inconsistency above-mentioned: and as it is impossible to suggest any other cause than an omission in calculation, he resumes the algebraical operations of No. 2, carefully retaining every part of the expression concerned. In this manner he finds a term multiplied by the evanescent factor $r^2 - a^2$; and having valued this term, in No. 7, he arrives at the true equation $\frac{1}{2} s + a \left(\frac{ds}{dr}\right) = -2\pi a^2 . v$.

From the account of that part of LAGRANGE's memoir which we have already examined, it is impossible to deny that the method of reasoning employed in LAPLACE's demonstration leads directly to the formula $\frac{1}{2} s + a \left(\frac{ds}{dr}\right) = o$; which nevertheless, in a particular case, is proved by LAGRANGE to be a false equation, the true one being $\frac{1}{2} s + a \left(\frac{ds}{dr}\right) = -2\pi a^2 . v$. Nor can it be controverted that the real reason of this difficulty, or rather error, is the omission of quantities which are indeed multiplied by the evanescent factor $r - a$, but which are not on that account, equal to nothing. In so far therefore the investigations of LAGRANGE coincide entirely with the conclusions obtained in my paper: and in effect the method of analysis which he employs does not differ materially from that made use of in No. 6 of my paper.

2. Let us now consider what is said in No. 4 and 5 of

LAGRANGE's memoir. By taking the fluxion of $\frac{1}{f}$, making r only variable, we get

$$r\left(\frac{d \cdot \frac{1}{f}}{dr}\right) = \frac{-(r^2 - ra \cdot \gamma)}{(r^2 - 2ra \cdot \gamma + a^2)^{\frac{3}{2}}} = (\text{when } r = a) - \tfrac{1}{2} \cdot \frac{r}{f}:$$

therefore $\frac{1}{2} \cdot \frac{1}{f} + a\left(\frac{d \cdot \frac{1}{f}}{dr}\right) = o$; which must be an identical equation; or such a one as, being expanded into a series of the powers of γ, will consist of terms that mutually destroy one another. Now since

$$\tfrac{1}{2} s + a\left(\frac{ds}{dr}\right) = \alpha \cdot a^2 \iint \left\{\tfrac{1}{2} \cdot \frac{1}{f} + a\left(\frac{d \cdot \frac{1}{f}}{dr}\right)\right\} \cdot v' \cdot d\mu' \cdot d\varpi';$$

we ought to have $\frac{1}{2} s + a\left(\frac{ds}{dr}\right) = o$: because the fluent may be considered as the sum of all the successive values of the fluxion, and an aggregate of nothings ought to be equal to nothing. This is the principle of LAPLACE's demonstration stated abstractly: and it cannot be exact; for LAGRANGE has proved that it fails in a particular case, and this failure he calls a paradox in the integral calculus.

LAGRANGE has actually reduced the function $\frac{1}{2} \cdot \frac{1}{f} + a\left(\frac{d \cdot \frac{1}{f}}{dr}\right)$ into a series; but as this would not assist us in solving the difficulty it needs not be noticed. In the preceding operations the supposition of $r = a$ has made all the terms containing the factor $r - a$ disappear, which it is nevertheless necessary to retain. For this purpose resume the formula set down above, viz.

$$r\left(\frac{d \cdot \frac{1}{f}}{dr}\right) = \frac{-(r^2 - ra \cdot \gamma)}{(r^2 - 2ra \cdot \gamma + a^2)^{\frac{3}{2}}};$$

and, because $r^2 - ra \cdot \gamma = \frac{1}{2}f^2 + \frac{r^2 - a^2}{2}$, we get

$$\frac{1}{2} \cdot \frac{1}{f} + r\left(\frac{d \cdot \frac{1}{f}}{dr}\right) = -\frac{1}{2} \cdot \frac{r^2 - a^2}{(r^2 - 2ra \cdot \gamma + a^2)^{\frac{3}{2}}}.$$

The last equation certainly proves that, when $r = a$, the function $\frac{1}{2} \cdot \frac{1}{f} + a\left(\frac{d \cdot \frac{1}{f}}{dr}\right)$ is evanescent for every value of γ, between the limits $+1$ and -1, with the single exception of the case $\gamma = 1$, when the function is infinitely great: and I shall now shew that it is to the overlooking of this last mentioned circumstance that all the difficulty and paradox attending this investigation have arisen.

Let it be proposed to find the value of the fluent $\int \frac{-(r-a) \cdot c \cdot dx}{\sqrt{r - ax}}$, between the limits $x = 1$ and -1. The indefinite fluent being $(r-a) \cdot \frac{2c}{a} \cdot \sqrt{r - ax}$, we get between the proposed limits,

$$\int \frac{-(r-a) \cdot c \cdot dx}{\sqrt{r - ax}} = (r-a) \cdot \frac{2c}{a} \cdot \left\{\sqrt{r+a} - \sqrt{r-a}\right\}.$$

This fluent is plainly $= o$, when $r = a$.

Let us next consider the fluent $\int \frac{-(r-a) \cdot c \cdot dx}{(r - ax)^2}$, between the same limits as before. Here the indefinite fluent being $-\frac{c}{a} \cdot \frac{r-a}{r-ax}$, we get between the proposed limits,

$$\int \frac{-(r-a) \cdot c \cdot dx}{(r-ax)^2} = \frac{c}{a} - \frac{c}{a} \cdot \frac{r-a}{r+a}.$$

In this instance the fluent does not vanish when $r = a$; for it is equal to $\frac{c}{a}$.

Lastly, let the fluent $\int \frac{-(r-a) \cdot c \cdot dx}{(r-ax)^3}$ be proposed. In this case, the indefinite fluent being $-\frac{c}{2a} \cdot \frac{r-a}{(r-ax)^2}$, we get between

the limits $+1$ and -1,

$$\int \frac{-(r-a) \cdot c \cdot dx}{(r-ax)^3} = \frac{c}{2a} \cdot \left\{ \frac{1}{r-a} - \frac{r-a}{(r+a)^2} \right\}.$$

And in this instance, the fluent is infinitely great when $r = a$.

All the three fluents which we have just been considering, ought to be alike equal to nothing, according to the reasoning of LAPLACE. For, when $r = a$, all the fluxions are evanescent for every value of x between the proposed limits, with the exception of the single case $x = 1$ in the two last, for which value of x the fluxions are infinitely great. And even in the first instance if we change the factor $r - a$ into $(r - a)^{\frac{m}{n}}$, making $\frac{m}{n}$ less than $\frac{1}{2}$; then in this case also the fluxion will be infinitely great when $x = 1$, while the whole fluent will still be evanescent as before. If therefore we would have an unerring criterion to direct us in such instances, we must consider the expression of the fluent. If that expression is finite at both the limits, and likewise for every intermediate value of the flowing quantity, then, on account of the evanescent factor, the whole integral will be equal to nothing: but if that expression becomes infinitely great at either of the limits, or for any intermediate value of the flowing quantity, then the whole fluent will be equal to a finite quantity when the evanescent factor is raised to the same power in the numerator and denominator; and it will be infinitely great, when the evanescent factor is raised to a higher power in the denominator than in the numerator. The examples we have given above fall under these three cases, and they are quite analogous to the distinction of cases in LAPLACE's theorem, as noticed in No. 6 of my paper. We may add farther that the

whole fluent will likewise be equal to nothing, when the evanescent factor is raised to a higher power in the numerator than in the denominator.

The explication here given is sufficient to clear up the paradox of Lagrange; and it certainly proves the inconclusive nature of Laplace's demonstration. One more remark is suggested by what has been said: the theorem of the last mentioned geometer is investigated by means of the direct method of fluxions alone, whereas the rules of the integral calculus are required in order to make the process rigorous and exact.

3. Lagrange having, in No. 7, obviated the difficulty in regard to the particular case when the thickness of the molecules spread over the surface of the sphere is a constant quantity, proceeds, in No. 8, to consider the general case when the thickness of the molecules is any function of the sines and cosines of the angles θ' and ϖ' that determine the position of a molecule with regard to a fixt pole on the surface of the sphere. In this case also the equation in the *Mécanique Céleste*, viz.

$$\tfrac{1}{2}V + a\left(\frac{dV}{dr}\right) = -\frac{2\pi a^2}{3}$$

cannot be exact, unless the equation

$$\tfrac{1}{2}s + a\left(\frac{ds}{dr}\right) = -2\pi a^2 . v$$

be proved to be true instead of the equation $\frac{1}{2}s + a\left(\frac{ds}{dr}\right) = o$ which would result from the demonstration of Laplace.

It must be recollected that

$$s = \alpha a^3 . \iint \frac{v' . d\mu' . d\varpi'}{\sqrt{r^2 - 2ra . \gamma + a^2}}$$

and that v is the same function of the sines and cosines of the

constant angles θ and ϖ, which determine the position of the attracted point in the surface of the sphere, that v' is of the variable angles θ' and ϖ'; in other words, v is what v' becomes at the attracted point. Perspicuity requires that we distinguish two cases: the first is, when v' is a rational and integral function of $\cos.\ \theta'$, $\sin.\ \theta' \cos.\ \varpi'$, $\sin.\ \theta' \sin.\ \varpi'$; or, μ', $\sqrt{1-\mu'^2}\ .\ \cos.\ \varpi'$, $\sqrt{1-\mu'^2}\ .\ \sin.\ \varpi'$; the second is, when v' is any other function of the sines and cosines of the angles θ' and ϖ'.

In the first case, LAGRANGE transforms v' into a function of γ, $\sqrt{1-\gamma^2}\ .\ \cos.\ \varphi$, $\sqrt{1-\gamma^2}\ .\ \sin.\ \varphi$,* which transformation he shews to be always possible; and having substituted $d\gamma\ .\ d\varphi$ for $d\mu'\ .\ d\varpi'$, he integrates the formula $\int \frac{v'\ .\ d\mu'\ .\ d\varpi'}{f^3}$, as he had done in No. 7 for the case when v' is a constant quantity, by a method entirely similar to that employed in No. 3 of my paper: and hence he proves the truth of the equation

$$\tfrac{1}{2}\, s + a \left(\frac{ds}{dr}\right) = -\, 2\pi a^2\ .\ v,$$

when $r = a$. Thus then LAPLACE's theorem is rigorously proved for an extensive class of spheroids; and in this point also the investigations of LAGRANGE coincide with the conclusions obtained in my paper.

With regard to the general case, when v' is any function of the sines and cosines of the angles θ' and ϖ', it is not easy to discover what are the precise sentiments of LAGRANGE. From his saying that the formula $\int \frac{v'\ .\ d\gamma\ .\ d\varphi}{f^3}$ is always integrable when v' is a rational and integral function of μ', $\sqrt{1-\mu'^2}\ .\ \cos.\ \varpi'$, $\sqrt{1-\mu'^2}\ .\ \sin.\ \varpi'$, are we to understand that the method, which follows in No. 9, is to be confined exclusively to

* See No. 3 of the preceding paper.

this case? or, when he says that it is sufficient for the purpose he has in view to have reduced the integration of the formula $\frac{v' \,.\, d\mu' \,.\, d\varpi'}{f^3}$ to that of the formula $\frac{-v' \,.\, d\gamma \,.\, d\phi}{f^3}$, are we to understand that the method of No. 9 is to be extended to every case when y' is any function of the sines and cosines of the angles θ' and ϖ'?

If the former be LAGRANGE's meaning, then we must suppose it to have been his intention to pass over in silence the more general case of the question which does not come under the method of integration he employs.

On the other hand, if we are to suppose that LAGRANGE intended his demonstration to apply to the general case when v' is any function of the sines and cosines of the angles θ' and ϖ'; then it must be owned that this part of his investigation is directly at variance with the conclusion drawn in my paper, which limits the truth of LAPLACE's theorem to the single case when v' is a rational and integral function of three rectangular co-ordinates of a point in the surface of a sphere. But, even if this be the sense of LAGRANGE, it will be allowed that, in so nice a case, a proof, which proceeds upon a transformation that cannot be performed, is not very decisive: and the following argument seems to destroy all the evidence of the process when it is extended beyond the natural boundary. If we integrate $v'd\phi$, between the limits $\phi = o$ and $\phi = 2\pi$, and put $\int v'd\phi = 2\pi \,.\, \Gamma^{(o)}$; then we shall have, as in No. 3 of my paper,

$$\iint \frac{v' \,.\, d\gamma \,.\, d\phi}{f^3} = 2\pi \,.\, \int \frac{\Gamma^{(o)} \,.\, d\gamma}{f^3}:$$

but the method of integration there employed, which is the

same as that in No. 9 of LAGRANGE's memoir, becomes unsatisfactory and undeserving the name of proof, except when all the functions $\Gamma^{(o)}, \frac{d\Gamma^{(o)}}{d\gamma}, \frac{dd\Gamma^{(o)}}{d\gamma^2}$, &c. are finite quantities at both the limits, and likewise for every intermediate value of γ; which will not be the case unless υ' be a rational and integral function of μ', $\sqrt{1-\mu'^2} \,.\, \cos. \varpi'$, $\sqrt{1-\mu'^2} \,.\, \sin. \varpi'$. Luckily, however, the author's own formulas suggest a clear and satisfactory way of determining this point without any transformation or the help of difficult integrations.

LAGRANGE has proved in the most incontestible manner, that the theorem of LAPLACE cannot be true unless the following equation likewise take place, viz.

$$\tfrac{1}{2}\, s + a \left(\frac{ds}{dr}\right) = -\, 2\pi a^2 \,.\, \upsilon :$$

and hence, it is plain, we shall be able to discover what function υ is of the sines and cosines of the angles θ and ϖ, by considering in what manner these quantities enter into the equivalent expression on the left-hand side. Let $x = \cos. \theta = \mu$, $y = \sqrt{1-\mu^2} \,.\, \cos. \varpi$, $z = \sqrt{1-\mu^2} \,.\, \sin. \varpi$; $x' = \cos. \theta' = \mu'$, $y' = \sqrt{1-\mu'^2} \,.\, \cos. \varpi'$, $z' = \sqrt{1-\mu'^2} \,.\, \sin. \varpi'$; then* $\gamma = \mu\mu' + \sqrt{1-\mu^2} \,.\, \sqrt{1-\mu'^2} \,.\, \cos. (\varpi' - \varpi) = xx' + yy' + zz'$; and, by substitution, we shall get

$$\frac{1}{f} = \frac{1}{\sqrt{r^2 - 2ra \,.\, (xx' + yy' + zz') + a^2}} \,.$$

therefore $\frac{1}{f}$ is a function of x, y, z; and $\frac{1}{2} \,.\, \frac{1}{f} + a \left(\frac{d \,.\, \frac{1}{f}}{dr}\right)$ will likewise be a function of the same quantities: but

$$\tfrac{1}{2} \,.\, s + a \left(\frac{ds}{dr}\right) = \iint . \left\{ \tfrac{1}{2} \,.\, \frac{1}{f} + a \left(\frac{d \,.\, \frac{1}{f}}{dr}\right) \right\} . \upsilon' \,.\, d\mu' \,.\, d\varpi' ;$$

* See No. 3 of my paper.

and because x, y, z are constant quantities, there will be the same powers and combinations of them in the integral as in the fluxion, the coefficients merely being changed: therefore the expression $\frac{1}{2}s + a\left(\frac{ds}{dr}\right)$ is likewise a function of x, y, z; and farther it is such a function as, being expanded into a series, can coincide only with a rational and integral function of the same quantities, consisting of a finite or infinite number of terms. Therefore the equation

$$\frac{1}{2}s + a\left(\frac{ds}{dr}\right) = -2\pi a^2 . v$$

cannot take place unless v is a like function of x, y, z.

The review which we have here taken of Lagrange's memoir, and the observations we have made upon it, confirm the conclusions drawn in my paper, and throw additional light upon this difficult subject. We are indebted to the skill and abilities of Laplace for the invention of an equation in partial fluxions which has already contributed much to advance our knowledge of that branch of physical astronomy which relates to the figure of the planets, and which promises still greater improvements by suggesting new methods and removing the obstacles that have impeded the researches of former mathematicians: but he has not been so happy in founding his application of this invention on the theorem concerning the attractions at the surfaces of spheroids. It is impossible to deny that this theorem, as it is delivered in the *Mécanique Céleste*, is unsupported by any demonstrative proof; and that the extent of it has not been well understood. Instead of the indirect investigation which Laplace has followed, it were to be wished, for the sake of greater clearness and of avoiding the subtilties that occur in his analysis, that the attractions of

such spheroids, as have been shown to fall under his method, were deduced directly from the function which expresses their radii: and on this account some degree of consideration may perhaps be attached to another paper of mine, presented to the Society about the middle of July last, in which an attempt is made to accomplish the object here mentioned.

Oct. 30, 1811.

II. *On the Attractions of an extensive Class of Spheroids. By* J. Ivory, *A. M.* *Communicated by* Henry Brougham, *Esq.* *F. R. S. M. P.*

Read November 14, 1811.

In this discourse I propose to investigate the attractions of a very extensive class of spheroids, of which the general description is, that they have their radii expressed by rational and integral functions of three rectangular co-ordinates of a point in the surface of a sphere. Such spheroids may be characterized more precisely in the following manner: conceive a sphere of which the radius is unit, and three planes intersecting one another at right angles in the centre; from any point in the surface of the sphere draw three perpendicular co-ordinates to the fixed planes, and through the same point in the surface likewise draw a right line from the centre, and cut off from that line a part equal to any rational and integral function of the three co-ordinates: then will the extremity of the part so cut off be a point in the surface of a spheroid of the kind alluded to; and all the points in the same surface will be determined by making the like construction for every point in the surface of the sphere. The term of a rational and integral function is not to be strictly confined here to such functions only as consist of a finite number of terms; it may

include infinite serieses, provided they are converging ones; and it may even be extended to any algebraic expressions that can be expanded into such serieses. This class of spheroids comprehends the sphere, the ellipsoid, both sorts of elliptical spheroids of revolution, and an infinite number of other figures, as well such as can be described by the revolving of curves about their axes, as others which cannot be so generated.

In the second chapter of the third book of the *Mécanique Céleste*, LAPLACE has treated of the attractions of spheroids of every kind; and in particular he has given a very ingenious method for computing the attractive forces of that class which in their figures approach nearly to spheres. In studying that work, I discovered that the learned author had fallen into an error in the proof of his fundamental theorem; in consequence of which he has represented his method as applicable to all spheroids whatever, provided they do not differ much from spheres; whereas in truth, when the error of calculation is corrected, and the demonstration made rigorous, his analysis is confined exclusively to that particular kind, described above, which it is proposed to make the subject of this discourse. I have already treated of this matter in a separate paper, in which I have pointed out the source of LAPLACE's mistake, and likewise have strictly demonstrated his method for the instances that properly fall within its scope. In farther considering the same subject, it occurred to me that the investigation in the second chapter of the third book of the *Mécanique Céleste*, however skilfully and ingeniously conceived, is nevertheless indirect, and is besides liable to another objection of

still greater weight; it does not exhibit the several terms of the series for the attractive force in separate and independent expressions: it only points out in what manner they may be derived successively, one after another; in so much that the terms of the series near the beginning cannot be found without previously computing all the rest. This remark gave occasion to the following paper, in which it is my design to give a solution of the problem which is not chargeable with the imperfections just mentioned: the analysis is direct, and every term of the series for the attractive force is deduced immediately from the radius of the spheroid. As the ellipsoid, which comprehends both sorts of elliptical spheroids of revolution, falls within the class of figures here treated of, I have derived, as a corollary from my investigation, the formulas for the attractions of that figure which are required in the theory of the earth: this paper therefore will contain all that is useful on the subject of the attractions of spheroids, as far as our knowledge at present extends, deduced by one uniform mode of analysis.

Having mentioned the principal object of this discourse, I must likewise notice a subordinate purpose I have in view; it is to put in a clear light the real grounds of LAPLACE's method, and of the equivalent method delivered in the following pages; to the accomplishment of which nothing is likely to contribute so much, as a direct and rigorous analysis perspicuously conducted. To promote the same end still farther, by preserving greater order and perspicuity in treating a subject in its own nature very complicated, this paper will be divided into two principal sections: in the first section it is proposed

to lay down the analytical propositions on which the investigation is founded: the second section will contain the solution of the problem under consideration.

One more preliminary observation it is proper to add. The problem of attractions contains two cases; when the density of the attracting body is uniform throughout; when it varies according to any given law: it is in the first of these two cases that the chief difficulties occur; and as I have nothing new to add on the second case, I shall here confine my attention to homogeneous spheroids, unit being supposed to denote the density.

I.

Preliminary Investigations.

1. Let μ denote the cosine of an angle, and let

$$f = \left\{ r^2 - 2ra.\mu + a^2 \right\}^{\frac{1}{2}};$$

then the truth of the following equation in partial fluxions will be proved merely by performing the operations indicated, viz.

$$(1-\mu^2)^n . \left(\frac{d . \frac{a - r\mu}{f^{2n+3}}}{dr} \right) + \left(\frac{d . \frac{(1-\mu^2)^{n+1}}{f^{2n+3}}}{d\mu} \right) = 0.$$

Now put $S = \frac{1}{f^{2n+1}}$; then

$$\left. \begin{aligned} \left(\frac{dS}{da}\right) &= -(2n+1) . \frac{a - r\mu}{f^{2n+3}} \\ \frac{1}{ra} . \left(\frac{dS}{d\mu}\right) &= 2n+1 . \frac{1}{f^{2n+3}} \end{aligned} \right\}$$

therefore, on account of the first equation, we shall obtain by substitution,

$$-ra \,.\, (1-\mu^2)^n \,.\, \left(\frac{ddS}{dadr}\right) + \left\{\frac{d\,.\left\{(1-\mu^2)^{n+1}\,.\left(\frac{dS}{d\mu}\right)\right\}}{d\mu}\right\} = 0.$$

2. Let $\frac{1}{f}$ be reduced into a series of the descending powers of r; then

$$\frac{1}{f} = C^{(0)} \,.\, \frac{1}{r} + C^{(1)} \,.\, \frac{a}{r^2} + C^{(2)} \,.\, \frac{a^2}{r^3} \ldots\ldots + C^{(i)} \,.\, \frac{a^i}{r^{i+1}}, \text{ \&c.}$$

and $C^{(i)}$ will be a rational and integral function of μ of i dimensions: substitute this series for S in the equation last found (n being $= 0$), and we shall obtain

$$i\,(i+1)\,C^{(i)} + \frac{d\,.\left\{(1-\mu^2)\,.\,\frac{dC^{(i)}}{d\mu}\right\}}{d\mu} = 0 \ldots\ldots (1).$$

Again, take the fluxions n times successively in $\frac{1}{f}$ and likewise in the series equivalent to it, making μ the only variable; and we shall get

$$\frac{1}{f^{2n+1}} = \frac{1}{1.3.5\ldots 2n-1} \,.\, \times \left\{\frac{1}{r^{2n+1}} \,.\, \frac{d^nC^{(n)}}{d\mu} + \frac{a}{r^{2n+2}} \,.\, \frac{d^nC^{(n+1)}}{d\mu^n} \ldots\ldots\ldots \right.$$

$$\left. + \frac{a^{i-n}}{r^{i+n+1}} \,.\, \frac{d^nC^{(i)}}{d\mu^n}, \text{ \&c.}\right\}:$$

substitute this series for S in the equation of No. 1, and we shall get

$$(i-n)\,(i+n+1)\,.\,(1-\mu^2)^n\,.\,\frac{d^nC^{(i)}}{d\mu^n} + \frac{d\,.\left\{(1-\mu^2)^{n+1}\,.\,\frac{d^{n+1}C^{(i)}}{d\mu^{n+1}}\right\}}{d\mu}$$

$$= 0 \ldots\ldots\ldots (2).$$

From this last equation it follows that

$$\int (1-\mu^2)^n \,.\, \frac{d^nC^{(i)}}{d\mu^n} \,.\, d\mu = 0$$

when the fluent is taken between the limits $\mu = -1$ and

$\mu = 1$: for the fluent in question is equal to $-\frac{1}{i-n \,.\, i+n+1} \,.$ $(1-\mu^2)^{n+1} \,.\, \frac{d^{n+1}C^{(i)}}{d\mu^{n+1}}$, a quantity which is evanescent at both the limits.

If we consider $\frac{d^{o}C^{(i)}}{d\mu^{o}}$ as a symbolical representation of $C^{(i)}$, the equation (1) will be included in the equation (2); whence it is easy to infer that whatever is proved of $\frac{d^{n}C^{(i)}}{d\mu^{n}}$ by the help of the equation (2) may be transferred to $C^{(i)}$ by putting $n = o$; a remark that will enable us to consult brevity, and of which we shall freely avail ourselves.

3. It is now proposed to find the value of $C^{(i)}$ in a series of the powers of μ.* The equation (2), by expanding its last term, will become

$$i\,(i+1)\,C^{(i)} - 2\mu \,.\, \frac{dC^{(i)}}{d\mu} + (1-\mu^2) \,.\, \frac{ddC^{(i)}}{d\mu^2} = o:$$

let the series

$$A^{(o)}\,\mu^{i} + A^{(1)}\,\mu^{i-2} + A^{(2)}\,\mu^{i-4} \,....\, + A^{(s)} \,.\, \mu^{i-2s} \,...\, + \,\&c.$$

be assumed as equivalent to $C^{(i)}$; then by substituting and equating the coefficient of μ^{i-2s} to o, we shall get

$$A^{(s)} = -\frac{(i-2\,s+2)\;(i-2\,s+1)}{2s\,(2i-2\,s+1)} \,.\, A^{(s-1)};$$

and, by putting $s = 1$, $s = 2$, &c. successively, we shall hence be able to determine the proportions of all the coefficients to the first one $A^{(o)}$, which must be investigated from other considerations. Now $C^{(i)}$ is the coefficient of $\frac{a^{i}}{r^{i+1}}$ in the ex-

* Méc. Céleste Liv. 3e, No. 15.

pansion of $\frac{1}{f} = \frac{1}{(r^2+a^2)^{\frac{1}{2}}} \cdot \left(1 - \frac{2ra \cdot \mu}{r^2+a^2}\right)^{-\frac{1}{2}}$; and, by the binomial theorem, the term containing μ^i will be $= \frac{1.3.5 \ldots 2i-1}{2.4.6 \ldots 2i} \times \frac{2^i r^i a^i \mu^i}{(r^2+a^2)^{i+\frac{1}{2}}} = \frac{1.3.5 \ldots 2i-1}{1.2.3 \ldots i} \cdot \frac{a^i \cdot \mu^i}{r^{i+1}} \cdot \left(1 + \frac{a^2}{r^2}\right)^{-(i+\frac{1}{2})}$; whence it is plain that $A^{(o)} = \frac{1.3.5 \ldots 2i-1}{1.2.3 \ldots i}$: consequently,

$$C^{(i)} = \frac{1.3.5 \ldots 2i-1}{1.2.3 \ldots i} \times \left\{\mu^i - \frac{i(i-1)}{2(2i-1)} \cdot \mu^{i-2} + \frac{i(i-1)(i-2)(i-3)}{2.4.(2i-1)(2i-3)} \cdot \mu^{i-4} - \&c.\right\}$$

If we take the fluxions n times successively in the last formula, we shall obtain

$$\frac{d^n C^{(i)}}{d\mu^n} = \frac{1.3.5 \ldots 2i-1}{1.2.3 \ldots i-n} \cdot \left\{\mu^{i-n} - \frac{i-n \cdot i-n-1}{2.2i-1} \cdot \mu^{i-n-2} + \frac{i-n \cdot i-n-1 \cdot i-n-2 \cdot i-n-3}{2.4 \cdot 2i-1 \cdot 2i-3} \cdot \mu^{i-n-4} - \&c.\right\}.$$

When $i-n$ is an even number, $\frac{d^n C^{(i)}}{d^n}$ will contain a part, equal to

$$\pm \frac{1.3.5 \ldots i+n+1}{2.4.6 \ldots i-n},$$

independent of μ; and when $i-n$ is an odd number, the same quantity will contain a part, equal to

$$\pm \frac{1.3.5 \ldots i+n}{2.4.6 \ldots i-n-1} \cdot \mu,$$

multiplied by μ only: these two parts of the value of $\frac{d^n C^{(i)}}{d\mu^n}$ we shall afterwards have occasion to refer to.

4. It is proposed to investigate the fluent of

$$(1-\mu^2)^n \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu,$$

between the limits $\mu = -1$ and $\mu = 1$; supposing P to be a rational and integral function of μ.

On account of the equation (2), we get

$$\int (1-\mu^2) \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu = -\frac{1}{i-n \cdot i+n+1} \cdot \int P \cdot d \cdot \left\{ (1-\mu^2)^{n+1} \cdot \frac{d^{n+1}C^{(i)}}{d\mu^{n+1}} \right\};$$

and, by integrating by parts,

$$\int (1-\mu^2)^n \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu = -\frac{1}{i-n \cdot i+n+1} \cdot (1-\mu^2)^{n+1} \cdot \frac{d^{n+1}C^{(i)}}{d\mu^{n+1}} \cdot P + \frac{1}{i-n \cdot i+n+1} \cdot \int (1-\mu^2)^{n+1} \cdot \frac{d^{n+1}C^{(i)}}{d\mu^{n+1}} \cdot \frac{dP}{d\mu} \cdot d\mu:$$

and, by rejecting that part of the fluent which is evanescent at both the limits, we have

$$\int (1-\mu^2)^n \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu = \frac{1}{i-n \cdot i+n+1} \cdot \int (1-\mu^2)^{n+1} \cdot \frac{d^{n+1}C^{(i)}}{d\mu^{n+1}} \cdot \frac{dP}{d\mu} \cdot d\mu.$$

In this last equation the expressions on both sides are entirely similar; and therefore by a repetition of the same operations we shall obtain

$$\int (1-\mu^2)^{n+1} \cdot \frac{d^{n+1}C^{(i)}}{d\mu^{n+1}} \cdot \frac{dP}{d\mu} \cdot d\mu = \frac{1}{i-n-1 \cdot i+n+2} \cdot \int (1-\mu^2)^{n+2} \cdot \frac{d^{n+2}C^{(i)}}{d\mu^{n+2}} \cdot \frac{d^2P}{d\mu^2} \cdot d\mu:$$

and exterminating the integral common to both these equations, we shall get

$$\int (1-\mu^2)^n \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu = \frac{1}{i-n \cdot i-n-1} \times \frac{1}{i+n+1 \cdot i+n+2} \times \int (1-\mu^2)^{n+2} \cdot \frac{d^{n+2}C^{i}}{d\mu^{n+2}} \cdot \frac{d^2P}{d\mu^2} \cdot d\mu.$$

It is evident we may continue the like operations as far as we

please: for abridging expressions let

$$\sigma = i-n \,.\, i-n-1 \,.\, i-n-2 \,\ldots\ldots\, i-n-m+1;$$

$$\tau = i+n+1 \,.\, i+n+2 \,.\, i+n+3 \,\ldots\ldots\, i+n+m;$$

then after m successive operations we shall get,

$$\int (1-\mu^2)^n \,.\, \frac{d^n C^i}{d\mu^n} \,.\, \mathrm{P} \,.\, d\mu = \frac{1}{\sigma . \tau} \times \int (1-\mu^2)^{n+m} \,.\, \frac{d^{n+m} C^{(i)}}{d\mu^{n+m}} \,.\, \frac{d^m \mathrm{P}}{d\mu^m} \,.\, d\mu.$$

If m, less then $i-n$, denote the dimensions of P, then $\frac{d^m \mathrm{P}}{d\mu^m}$ will be a constant quantity, and the fluent on the right-hand side will be $= o$ (No. 2): hence this theorem, viz.

"If P be a rational and integral function of μ, and of less "dimensions than $i-n$, then

$$\int (1-\mu^2)^n \,.\, \frac{d^n C^{(i)}}{d\mu^n} \,.\, \mathrm{P} \,.\, d\mu = o$$

"when the whole fluent is taken between the limits $\mu = -1$ "and $\mu = 1$."

If the dimensions of P be not less than $i-n$, put $m = i-n$, and for $\frac{d^i C^{(i)}}{d\mu^i}$ write its value, $1.3.5 \ldots . 2i-1$ (3); and the preceding formula will become

$$\int (1-\mu^2)^n \,.\, \frac{d^n C^{(i)}}{d\mu^n} \,.\, \mathrm{P} \,.\, d\mu = \frac{i-n+1 \,.\, i-n+2 \,\ldots\ldots\, i+n}{2.4.6 \,\ldots\ldots\ldots\ldots\, 2i} \times \int (1-\mu^2)^i \,.\, \frac{d^{i-n} \mathrm{P}}{d\mu^{i-n}} \,.\, d\mu:$$

and hence, $\beta^{(n)} = \frac{1}{i-n+1 \,.\, i-n+2 \,\ldots\ldots\, i+n}$, we have

$$\beta^{(n)} \,.\int (1-\mu^2)^n \,.\, \frac{d^n C^{(i)}}{d\mu^n} \,.\, \mathrm{P} \,.\, d\mu = \frac{\int (1-\mu^2)^i \,.\, \frac{d^{i-n} \mathrm{P}}{d\mu^{i-n}} \,.\, d\mu}{2.4.6 \,\ldots\ldots\ldots\ldots\, 2i} \,.$$

By means of the last formula the fluent in question will be reduced to the integration of expressions of this kind, viz. $\mu^{s-1} \cdot (1-\mu^2)^i \cdot d\mu$; a research with which mathematicians are familiar. In the first place when s is even; then, considering the definite fluent between the limits $\mu = -1$ and $\mu = 1$; we have

$$\int \mu^{s-1} \cdot (1-\mu^2)^i \cdot d\mu = o:$$

and indeed, supposing P to be any odd function of μ, we have more generally $\int P \cdot d\mu = o$, between the same limits. In the second place when s is odd; then, taking the definite fluent as before,

$$\int \mu^{s-1} \cdot (1-\mu^2)^i \cdot d\mu = \frac{2}{s} \cdot \frac{2i}{2i+s} \cdot \frac{2(i-1)}{2i+s-2} \cdot \frac{2(i-3)}{2i+s-4} \dots\dots \frac{2}{2+s}.$$

The observations that have already been made are sufficient to point out in what manner the expressions of the fluents under consideration may be formed with great practical commodiousness.

5. Let μ, μ', γ denote the cosines of the three sides of a spherical triangle; and let ϕ be the angle opposite to the side whose cosine is γ: then, according to what is taught in spherical trigonometry,

$$\gamma = \mu\mu' + \sqrt{1-\mu^2} \cdot \sqrt{1-\mu'^2} \cdot \cos.\ \phi:$$

suppose farther that $f = \left\{r^2 - 2ra \cdot \gamma + a^2\right\}^{\frac{1}{2}}$, and let

$$\frac{1}{f} = Q^{(0)} \cdot \frac{1}{r} + Q^{(1)} \cdot \frac{a}{r^2} + Q^{(2)} \cdot \frac{a^2}{r^3} \dots\dots + Q^{(i)} \cdot \frac{a^i}{r^{i+1}} \ \&c.$$

it is required to expand $Q^{(i)}$, which is the same function of γ that $C^{(i)}$ is of μ, into a series of the cosines of ϕ and its multiples.*

* Méc. Cél. Liv. 3e, No. 15..

LAPLACE has proved that every one of the coefficients in the series for $\frac{1}{f}$ will satisfy an equation in partial fluxions which is thus generally expressed for $Q^{(i)}$, viz.

$$i(i+1)\cdot Q^{(i)} + \left\{\frac{d\cdot\left\{(1-\mu^2)\cdot\left(\frac{dQ^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\} + \frac{\left(\frac{ddQ^{(i)}}{d\varphi^2}\right)}{1-\mu^2} = 0.$$

This is a fundamental equation in his investigation, and it is necessary for effecting the expansion here proposed: but we shall refer to LAPLACE's work for the demonstration of it.*

It is plain that $Q^{(i)}$, when it is considered as a function of μ and the cosines of φ and its multiples, may be thus represented, viz.

$Q^{(i)} = H^{(0)} + (1-\mu^2)^{\frac{1}{2}}\cdot H^{(1)}\cdot \cos.\varphi + (1-\mu^2)^{\frac{2}{2}}\cdot H^{(2)}\cdot$ $\cos. 2\varphi + \&c.$

the general term of the series being $(1-\mu^2)^{\frac{n}{2}}\cdot H^{(n)}\cdot\cos. n\varphi$, which ought to satisfy LAPLACE's equation in partial fluxions: now, having actually substituted that quantity in the equation mentioned, and having divided all the terms by $\cos. n\varphi$, I have found,

$$(i-n)(i+n+1)\cdot(1-\mu^2)^{\frac{n}{2}}\cdot H^{(n)} - 2(n+1)\mu(1-\mu^2)^{\frac{n}{2}}\cdot\frac{dH^{(n)}}{d\mu} + (1-\mu^2)^{\frac{n}{2}+1}\cdot\frac{ddH^{(n)}}{d\mu^2} = 0:$$

and, after having multiplied all the terms by $(1-\mu^2)^{\frac{n}{2}}$, the result will be equivalent to this equation, viz.

$$(i-n)(i+n+1)(1-\mu^2)^n\cdot H^{(n)} + \frac{d.\left\{(1-\mu^2)^{n+1}\cdot\frac{dH^{(n)}}{d\mu}\right\}}{d\mu} = 0,$$

whence it follows (equat. 2.) that $H^{(n)} = B^{(n)}\cdot\frac{d^nC^{(i)}}{d\mu^n}$, where

* Méc. Cél. No. 9, Liv. 3e, and No. 11, Liv. 2d.

$B^{(n)}$ denotes a quantity that does not contain μ; therefore the general term of the series for $Q^{(i)}$ is $B^{(n)} . (1-\mu^2)^{\frac{n}{2}} . \frac{d^nC^{(i)}}{d\mu^n}$. cos. $n\varphi$: but as μ and μ' enter alike into the expression of $Q^{(i)}$, it is clear that they will be both equally concerned in every term of its expansion: therefore the general term of the series will be,

$$\beta^{(n)} . (1-\mu^2)^{\frac{n}{2}} . (1-\mu'^2)^{\frac{n}{2}} . \frac{d^nC^{(i)}}{d\mu^n} . \frac{d^nC'^{(i)}}{d\mu'^n} . \cos. n\varphi,$$

where $C'^{(i)}$ is put to denote the same function of μ' that $C^{(i)}$ does of μ; and $\beta^{(n)}$ is a quantity that contains neither μ nor μ', and which can only be a numeral coefficient, and is all that now remains unknown.

In order to determine $\beta^{(n)}$, we must follow the process of LAPLACE.* It is to be observed that $Q^{(i)}$ is the coefficient of $\frac{a^i}{r^{i+1}}$ in the expansion of the radical $\left\{r^2 - 2ra . \gamma + a^2\right\}^{-\frac{1}{2}} =$ $\left\{r^2 - 2ra . (\mu\mu' + \sqrt{1-\mu^2} . \sqrt{1-\mu'^2} . \cos. \varphi) + a^2\right\}^{-\frac{1}{2}}$; which, when the squares and other higher powers are neglected, will be equal to

$\left\{r^2 - 2ra . \cos. \varphi + a^2\right\}^{-\frac{1}{2}} + ra . \mu\mu' . \left\{r^2 - 2ra . \cos. \varphi + a^2\right\}^{-\frac{3}{2}}$;

from the first term of this expression are derived all the parts of the expansion of the radical $\left\{r^2 - 2ra . \gamma + a^2\right\}^{-\frac{1}{2}}$ which are independent on μ and μ'; and from the second term of it are derived all those parts which contain only $\mu\mu'$, without the squares and higher powers: now if we determine the parts

* Méc. Cél. Liv. 3e, No. 15.

mentioned by the actual expansion of the two radicals, and likewise determine the corresponding parts of $Q^{(i)}$ by means of the formulas in No. 3; the comparison of the equivalent expressions will determine the values of the coefficients required.

To execute the operations alluded to, let c denote the number whose hyperbolic logarithm is unit; then

$$\left\{r^2 - 2ra \,.\, \cos.\, \phi + a^2\right\}^{-s} = \left(r - a \,.\, c^{\phi\sqrt{-1}}\right)^{-s} . \left(r - a \,.\, c^{-\phi\sqrt{-1}}\right)^{-s};$$

and if we represent the expansions of the two binomials by the serieses

$$\frac{1}{r^s} + A^{(1)} . \frac{a \,.\, c^{\phi\sqrt{-1}}}{r^{1+s}} + A^{(2)} . \frac{a^2 \,.\, c^{2\phi\sqrt{-1}}}{r^{2+s}} + \&c.$$

$$\frac{1}{r^s} + A^{(1)} . \frac{a \,.\, c^{-\phi\sqrt{-1}}}{r^{1+s}} + A^{(2)} . \frac{a^2 . c^{-2\phi\sqrt{-1}}}{r^{2+s}} + \&c.$$

we shall obtain the expansion of the radical by multiplying the two serieses: let p and q denote the ranks of any two terms in both serieses, then the part of the expansion derived from the multiplication of the aforesaid parts, will be

$$2A^{(p)} . A^{(q)} . \frac{a^{p+q}}{r^{p+q+2s}} . \left\{\frac{c^{(p-q)\,\phi\sqrt{-1}} + c^{-(p-q)\phi\sqrt{-1}}}{2}\right\};$$

$$\text{or, } 2A^{(p)} . A^{(q)} . \frac{a^{p+q}}{r^{p+q+2s}} . \cos.\, (p-q) . \phi.$$

When $i - n$ is an even number, we have only to make $p + q = i$, and $p - q = n$, and $s = \frac{1}{2}$; and we shall get

$$2 \times \frac{1.3.5..\,.i-n-1}{2.4.6.\ldots\; i-n} \times \frac{1.3.5.\ldots i+n-1}{2.4.6.\ldots\; i+n} . \cos.\, n\phi,$$

for the part of the coefficient of $\frac{a^i}{r^{i+1}}$, or of $Q^{(i)}$, which is mul-

tiplied by cos. $n\phi$ and clear of μ and μ'; but the like part of $\beta^{(n)} . (1-\mu^2)^{\frac{n}{2}} . (1-\mu'^2)^{\frac{n}{2}} . \frac{d^n C^{(i)}}{d\mu^n} . \frac{d^n C'^{(i)}}{d\mu^n}$. cos. $n\phi$ (which is the whole expression of the part of $Q^{(i)}$ multiplied by cos. $n\phi$) obtained by the help of the formulas in No. 3, is

$$\beta^{(n)} . \left(\frac{1.3.5 \ldots i+n-1}{2.4.6 \ldots i-n}\right)^2 . \cos. n\phi :$$

therefore by equating the equivalent expressions, we get

$$\beta^{(n)} = \frac{2}{i-n+1 . i-n+2 \ldots\ldots i+n} .$$

When $i-n$ is odd, make $p+q=i-1$; $p-q=n$; $s=\frac{3}{2}$: then we will obtain

$$2 \times \frac{1.3.5 \ldots i+n}{2.4.6 \ldots i+n-1} . \frac{1.3.5 \ldots i-n}{2.4.6 \ldots i-n-1} . \mu\mu' . \cos. n\phi$$

for the part of $Q^{(i)}$, or of the coefficient of $\frac{a^i}{r^{i+1}}$, which is multiplied by $\mu\mu'$. cos, $n\phi$: but the like part of $\beta^{(n)} . (1-\mu^2)^{\frac{n}{2}} . (1-\mu'^2)^{\frac{n}{2}} . \frac{d^n C^{(i)}}{d\mu^n} . \frac{d^n C'^{(i)}}{d\mu^n}$. cos. $n\phi$, obtained by the formulas in No. 3, is

$$\beta^{(n)} . \left(\frac{1.3.5 \ldots i+n}{2.4.6 \ldots i-n-1}\right)^2 . \mu\mu' . \cos. n\phi :$$

whence we get, in this case also,

$$\beta^{(n)} = \frac{2}{i-n+1 . i-n+2 \ldots\ldots i+n} .$$

Now if we write $2\beta^{(n)}$ in the place of $\beta^{(n)}$; that is, if we henceforth put (as in No. 4)

$$\beta^{(n)} = \frac{1}{i-n+1 . i-n+2 . i-n+3 \ldots\ldots i+n};$$

then all the terms of the expansion we are seeking, will be found by making $n=1$, $n=2$, $n=3$, &c. successively, and it will be thus expressed, viz.

$$
\begin{aligned}
Q^{(i)} = C^{(i)} \,.\, C'^{(i)} &+ 2\beta^{(1)} \,.\, (1-\mu^2)^{\frac{1}{2}} \,.\, (1-\mu'^2)^{\frac{1}{2}} \,.\, \frac{dC^{(i)}}{d\mu} \,.\, \frac{dC'^{(i)}}{d\mu'} \,.\, \cos.\ \phi \\
&+ 2\beta^{(2)} \,.\, (1-\mu^2)^{\frac{2}{2}} \,.\, (1-\mu'^2)^{\frac{2}{2}} \,.\, \frac{d^2C^{(i)}}{d\mu^2} \,.\, \frac{d^2C'^{(i)}}{d\mu'^2} \,.\, \cos.\ 2\phi \\
&\ \ \vdots \\
&+ 2\beta^{(n)} \,.\, (1-\mu^2)^{\frac{n}{2}} \,.\, (1-\mu'^2)^{\frac{n}{2}} \,.\, \frac{d^nC^{(i)}}{d\mu^n} \,.\, \frac{d^nC'^{(i)}}{d\mu'^n} \,.\, \cos.\ n\phi \\
&\quad \&c.
\end{aligned}
$$

II.

Investigation of the Attractions of Spheroids of a particular Kind.

6. Instead of seeking immediately the attraction of a spheroid in any proposed direction, it will be more advantageous to investigate (as Laplace has done) the value of the expression (to be henceforth denoted by V) which is the sum of the quotients produced by dividing all the molecules of the mass of the spheroid by their respective distances from the attracted point. For such is the nature of the analytical expression now mentioned, that if it be first transformed into a function of three rectangular co-ordinates one of which is parallel to a line given by position, and the fluxion with regard to this co-ordinate be taken; the coefficient of the partial fluxion after its sign is changed, will denote the attractive force which acts parallel to the given line. In order to demonstrate this property of the function V, we shall suppose that x, y, z denote the co-ordinates of the molecule $d\mathrm{M}$, and a, b, c, the co-ordinates of the attracted point: then

$$V = \int \frac{d\mathrm{M}}{\left\{(a-x)^2 + (b-y)^2 + (c-z)^2\right\}^{\frac{1}{2}}};$$

the fluent being understood to be extended to all the molecules of the mass of the spheroid: now if the fluxion of this

expression be taken, making a the only variable, we shall have

$$-\left(\frac{dV}{da}\right)=\int\frac{(a-x)\,.\,dM}{\left\{(a-x)^2+(b-y)^2+(c-z)^2\right\}^{\frac{3}{2}}}:$$

where the expression on the right hand side is the attractive force parallel to a, as will readily appear by decomposing the direct attractions of all the molecules into the partial attractions parallel to the co-ordinates. But, besides enabling us to find the attractive force in any proposed direction, the function V has another advantage; for it is this function, and not the expressions of the attractive forces, which enters into the equation of the surface of a body, wholly or partly fluid, in a state of equilibrium.*

The expression for V, exhibited above is not of a commodious form, and on this account it becomes necessary to transform it. Let $x = R' \cos.\theta'$; $y = R' \sin.\theta' \cos.\varpi'$; and $z = R' \sin.\theta' \sin.\varpi'$; then will R′ be the line drawn from the molecule dM to the origin of the co-ordinates; θ' will be the angle which R′ makes with the axis of x; and ϖ' the angle which the projection of R′ upon the plane to which x is perpendicular, makes with a line given by position in the same plane: from the assumed values of x, y, z, it is easy to derive these new values, viz.

$$x = R' \cos.\theta' = \sqrt{R'^2 - y^2 - z^2}$$
$$y = R' \sin.\theta' \cos.\varpi' = \sqrt{R'^2 \sin.^2\theta' - z^2}$$
$$z = R' \sin.\theta' \sin.\varpi':$$

and, by taking the fluxions so as to make x vary with R′, y with θ', and z with ϖ'; which will leave dx, dy, dz, as well as dR', $d\theta'$, $d\varpi'$, unrelated and independent on one another as the

* Méc. Cél. Liv. 3, No. 4.

case requires, we shall have

$$dx = \frac{R' dR'}{\sqrt{R'^2 - y^2 - z^2}} = \frac{dR'}{\cos.\theta'}$$

$$dy = \frac{R'^2 \sin.\theta' \cos.\theta'.d\theta'}{\sqrt{R'^2 \sin.^2\theta' - z^2}} = \frac{R' \cos.\theta'.d\theta'}{\cos.\varpi'}$$

$$dz = R' \sin.\theta' \cos.\varpi'.d\varpi':$$

consequently (the density being denoted by unit) $dM = dx.dy.dz = R'^2.dR'.d\theta' \sin.\theta'.d\varpi$. Farther, let $a = r\cos.\theta$, $b = r \sin.\theta \cos.\varpi$, $c = r \sin.\theta \sin.\varpi$; then, by substitution,

$$*\ V = \iiint \frac{R'^2.dR'.d\theta' \sin.\theta'.d\varpi'}{\sqrt{r^2 - 2rR'.(\cos.\theta \cos.\theta' + \sin.\theta \sin.\theta' \cos.(\varpi' - \varpi)) + R'^2}}$$

and if we put $\cos.\theta = \mu$, $\cos.\theta' = \mu'$; then

$$\left.\begin{array}{l} V = \iiint \dfrac{R'^2.dR'.d\mu'.d\varpi'}{\sqrt{r^2 - 2rR'.\gamma + R'^2}} \\ \gamma = \mu\mu' + \sqrt{1-\mu^2}.\sqrt{1-\mu'^2}.\cos.(\varpi' - \varpi) \end{array}\right\} (4).$$

7. When the attracted point is without the surface, the expression for V, in order to embrace the whole mass of the spheroid, must be integrated from $R' = o$, to $R' = R$, R denoting what R′ becomes at the surface; from $\mu' = -1$ to $\mu' = 1$; and from $\varpi' = o$ to $\varpi' = 2\pi$, 2π being the circumference when the radius is unit. In this case V must be reduced into a series containing the descending powers of r, which we may thus represent, viz.

$$V = \frac{B^{(o)}}{r} + \frac{B^{(1)}}{r^2} + \frac{B^{(2)}}{r^3} \ldots\ldots + \frac{B^{(i)}}{r^{i+1}} \ldots \&c.$$

and if we expand the radical in the last expression of V into a similar series, and use $Q^{(i)}$ to denote the same thing as formerly in No. 4, we shall get, by equating the corresponding terms,

* Méc. Cél. Liv. 3e, No. 8.

$$B^{(i)} = \iiint R'^{i+2} \cdot dR' \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}.*$$

In this expansion $B^{(0)}$, in every case is equal to the mass of the spheroid: and with regard to the second term, LAPLACE has remarked that it may be made to disappear by fixing the origin of R′, which is an arbitrary point, in the centre of gravity of the spheroid. To prove this, we have

$$B^{(1)} = \iiint R'^3 \cdot dR' \cdot d\mu' \cdot d\varpi' \cdot Q^{(1)}:$$

but $dM = R'^2 \cdot dR' \cdot d\mu' \cdot d\varpi'$; and $R' \cdot Q^{(1)} = R' \cdot \gamma = \mu \times R'\mu' + \sqrt{1-\mu^2} \cdot \cos.\varpi \times R' \cdot \sqrt{1-\mu'^2} \cdot \cos.\varpi' + \sqrt{1-\mu^2} \cdot \sin.\varpi \times R' \cdot \sqrt{1-\mu'^2} \cdot \sin.\varpi' = \mu \times x + \sqrt{1-\mu^2} \cdot \cos.\varpi \times y + \sqrt{1-\mu^2} \cdot \sin.\varpi \times z$; where x, y, z denote as before the coordinates of the molecule dM: therefore, by substitution,

$$B^{(1)} = \mu \times \int x \cdot dM + \sqrt{1-\mu^2} \cdot \cos.\varpi \times \int y \cdot dM$$
$$+ \sqrt{1-\mu^2} \cdot \sin.\varpi \times \int z \cdot dM:$$

now, if all the planes to which x, y, z, are perpendicular pass through the centre of gravity; then, by the nature of that point, $\int x \cdot dM = 0$; $\int y \cdot dM = 0$; $\int z \cdot dM = 0$: therefore $B^{(1)} = 0$.†

In the expression of $B^{(i)}$ none of the integrations can be executed in a general manner, excepting that relative to dR': let R denote what R′ becomes at the surface of the spheroid; then

$$B^{(i)} = \frac{1}{i+3} \cdot \iint R^{i+3} \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}.$$

8. When the attracted point is within the spheroid, the value of V will be represented by a series of the ascending powers of r: let

* Méc. Cél. Liv. 3e, No. 9. † Ibid. Liv. 3e, No. 12.

$$V = b^{(0)} + b^{(1)} . r + b^{(2)} . r^2 + b^{(3)} . r^3 + \&c.;$$

then by expanding the radical in the formula (4) into a series of a similar form, and equating the corresponding terms, we shall get

$$b^{(i)} = \iiint \frac{dR' . d\mu' . d\varpi' . Q^{(i)}}{R^{i-1}} .*$$

In this value of $b^{(i)}$, the integration with regard to dR' cannot be executed from $R' = o$, as in the former case; because this expansion of V necessarily supposes that the attracted point is included within all the attracting matter: let R be what R′ becomes at the surface of the spheroid, which is the outer surface bounding the attracting matter, and let ρ be the radius of the inner surface; then, with respect to the matter between the two surfaces, and for a point within them both, we shall have

$$b^{(i)} = \frac{1}{i-2} . \iint \left\{ \frac{1}{\rho^{i-2}} - \frac{1}{R^{i-2}} \right\} . d\mu' . d\varpi' . Q^{(i)} . (6).$$

In the case of $i = 2$, the expression of the coefficient takes a particular form: for

$$b^{(2)} = \iiint \frac{dR' . d\mu' . d\varpi' . Q^{(2)}}{R'};$$

and, by integrating,

$$b^{(2)} = \iint \left\{ \log. R' - \log. \rho \right\} . d\mu' . d\varpi' . Q^{(2)}.$$

Let us now seek an expression of the force with which the whole spheroid attracts a point within the surface. For this purpose we shall suppose ρ to denote the radius of a sphere which completely envelops the spheroid: and we shall determine; first, the value of V, relatively to the matter between the spheroid and the sphere; secondly, its value, relatively to

* Méc. Cél. Liv. 3e, No. 13.

the whole sphere: then the difference of these values will be the quantity proposed to be investigated.

With regard to the first value of V, it is to be observed that R is here the radius of the inner, and ρ that of the outer surface; therefore (6),

$$b^{(i)} = \frac{1}{i-2} \cdot \iint \left\{ \frac{1}{R^{i-2}} - \frac{1}{\rho^{i-2}} \right\} \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}.$$

But I say that $\int Q^{(i)} \cdot d\mu' \cdot d\varpi' = o$, when the fluent is extended between the proper limits: for μ, μ', and γ are the cosines of the three sides of a spherical triangle, and $\varpi' - \varpi$ is the angle of the same triangle opposite to the side whose cosine is γ; and if we put ψ to denote the angle opposite to the side whose cosine is μ'; then since the fluxion of the spherical surface may be either $d\mu' \cdot d\varpi'$ or $d\gamma \cdot d\psi$; therefore, when the fluents are extended to the whole surface of the sphere, we shall have

$$\int Q^{(i)} \cdot d\mu' \cdot d\varpi' = \int Q^{(i)} \cdot d\gamma \cdot d\psi = 2\pi \cdot \int Q^{(i)} \cdot d\gamma :$$

but $\int Q^{(i)} \cdot d\gamma$, between the limits $\gamma = -1$ and $\gamma = 1$, is $= o$ (No. 2): therefore $\int Q^{(i)} \cdot d\mu' \cdot d\varpi' = o$.

Consequently the preceding expression of $b^{(i)}$ will become simply

$$b^{(i)} = \frac{1}{i-2} \cdot \iint \frac{Q^{(i)} \cdot d\mu \cdot d\varpi'}{R^{i-2}} :$$

and the value of V, relative to the shell of matter between the spheroid and sphere will be expressed by this series, viz.

$$\begin{aligned} V = \tfrac{1}{2} \iint (\rho^2 - R^2) \cdot d\mu' \cdot d\varpi' - r \cdot \iint R \cdot d\mu' \cdot d\varpi' \cdot Q^{(1)} \\ - r^2 \cdot \iint \log. R \cdot d\mu' \cdot d\varpi' \cdot Q^{(2)} \\ + r^3 \cdot \iint \frac{Q^{(3)} \cdot d\mu' \cdot d\varpi'}{R^2} \\ + \&c. \end{aligned}$$

As to the value of V for the whole sphere, it is composed of two parts: one relative to the matter within the attracted point, which is a sphere whose radius is the distance of that point from the centre; and the other, relative to the remaining matter of the sphere: the value of the first part is $= \frac{4\pi}{3} \cdot r^2$; the value of the second part is $= \frac{1}{2}\iint (\rho^2 - r^2) \times d\mu' \cdot d\varpi'$: therefore the whole value of V is $= \frac{4\pi}{3} \cdot r^2 + \frac{1}{2}\iint (\rho^2 - r^2) \cdot d\mu' \cdot d\varpi'$.

By taking the difference of these two values, we get

$$\begin{aligned}
V = & - \frac{2\pi}{3} \cdot r^2 + \frac{1}{2} \cdot \iint R^2 \cdot d\mu' \cdot d\varpi' \cdot Q^{(0)} \\
& + r \cdot \iint R \cdot d\mu' \cdot d\varpi' \cdot Q^{(1)} \\
& + r^2 \cdot \iint \log. R \cdot d\mu' \cdot d\varpi' \cdot Q^{(2)} \\
& - \frac{r^3}{1} \cdot \iint \frac{Q^{(3)} \cdot d\mu' \cdot d\varpi'}{R} \\
& - \frac{r^4}{2} \cdot \iint \frac{Q^{(4)} \cdot d\mu' \cdot d\varpi'}{R^2} \\
& \&c.
\end{aligned}$$

this is the value of V when the attracted point is within the spheroid; and the terms in it that are unknown depend only on the radius of the surface, as in the case when the attracted point is without the surface.

9. We now proceed to the application of the formulas that have been investigated. And in the first place we shall consider a spheroid differing little from a sphere: in which case $R = a \cdot (1 + \alpha \cdot y')$, α denoting a coefficient so small that its square and other higher powers may be neglected; and y' a rational and integral function of μ', $\sqrt{1 - \mu'^2} \cdot \cos. \varpi'$ and

$\sqrt{1-\mu'^2}$. sin. ϖ'. It is to be understood that $a\,.\,(1+\alpha\,.\,y)$ denotes that radius of the spheroid which, produced if necessary, passes through the attracted point; and y is what y' becomes when $\mu'=\mu$ and $\varpi'=\varpi$.

Supposing the attracted point to be without the surface, we have No. 7,

$$\left.\begin{aligned} V &= \frac{B^{(0)}}{r} + \frac{B^{(1)}}{r^2} + \frac{B^{(2)}}{r^3} \ldots\ldots + \frac{B^{(i)}}{r^{i+1}} \ldots \&c. \\ B^{(i)} &= \frac{1}{i+3} \,.\, \iint R^{i+3} \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)}. \end{aligned}\right\}$$

and by substituting $a\,.\,(1+\alpha\,.\,y')$ for R and retaining only quantities of the first order with regard to α, we shall get,

$$B^{(i)} = \frac{a^{i+3}}{i+3} \,.\, \iint Q^{(i)} \,.\, d\mu' \,.\, d\varpi' + \alpha \,.\, a^{i+3} \,.\, \iint y' \,.\, d\mu' \,.\, d\varpi'\, Q^{(i)}:$$

but, as has already been proved (No. 8), $\iint Q^{(i)} \,.\, d\mu' \,.\, d\varpi' = o$: therefore

$$B^{(i)} = \alpha \,.\, a^{i+3} \,.\, \iint y' \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)}:$$

thus the value of $B^{(i)}$ depends upon the integral $\iint y' \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)}$, which may be found by means of the analytical formulas in the first part of this discourse, as we now proceed to show.

In the first place, when y' is a rational and integral function of μ' only without ϖ', which will be the case in spheroids of revolution: substitute for $Q^{(i)}$ its developement in No. 5, writing $\varpi'-\varpi$ for ϕ; integrate from $\varpi'=o$ to $\varpi'=2\pi$, observing that the fluents of all the terms which contain the cosines of $\varpi'-\varpi$ are of the same magnitude at both the limits, and therefore they will add nothing to the value of the integral taken between these limits: then we shall have simply

$$\iint y' . d\mu' . d\varpi' . Q^{(i)} = 2\pi \, C^{(i)} . \int y' . d\mu' . C'^{(i)} :$$

to execute the remaining integration we have only to apply the method of No. 4: let the integral $\iint y' . d\mu' . d\varpi' . Q^{(i)}$ be denoted by $2\pi \times U^{(i)}$; then by the method alluded to,

$$U^{(i)} = C^{(i)} \times \frac{\int (1-\mu'^2)^i . \frac{d^i y'}{d\mu'^i} . d\mu'}{2.4.6 \ldots\ldots\ldots 2i}.$$

If y' be a rational and integral function of μ', $\sqrt{1-\mu'^2}$. cos. ϖ' and $\sqrt{1-\mu'^2}$. sin. ϖ'; it must be transformed into a series of the sines and cosines of ϖ' and its multiples; then

$$y' = M^{(0)} + (1-\mu'^2)^{\frac{1}{2}} . M^{(1)} . \cos. \varpi' + (1-\mu'^2)^{\frac{2}{2}} . M^{(2)} . \cos. 2\varpi' \, \&c.$$
$$+ (1-\mu'^2)^{\frac{1}{2}} . N^{(1)} . \sin. \varpi' + (1-\mu'^2)^{\frac{2}{2}} . N^{(2)} . \sin. 2\varpi' \, \&c.$$

the general term of the series being $(1-\mu'^2)^{\frac{n}{2}} . M^{(n)} . \cos. n\varpi'$ $+ (1-\mu'^2)^{\frac{n}{2}} . N^{(n)} . \sin. n\varpi'$, where $M^{(n)}$ and $N^{(n)}$ denote rational and integral functions of μ'; and here the integral in question will consist of as many parts as there are independent functions contained in y'. In order to find the part of the integral resulting from the general term, we must multiply that term into the expansion of $Q^{(i)}$ investigated in No. 5; and in combining these two expressions we may omit all the terms which, after multiplication, would contain the sines and cosines of the multiples of ϖ'; because these, when they are integrated with regard to $d\varpi'$, will be of the same value at both the limits, on which account they will produce nothing in the value of the integral: this being observed, the only term of $Q^{(i)}$ which it is necessary to retain is that one containing cos. $n(\varpi' - \varpi)$, which may be thus written,

$$2\beta^{(n)} \cdot (1-\mu^2)^{\frac{n}{2}} \cdot (1-\mu'^2)^{\frac{n}{2}} \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot \frac{d^n C'^{(i)}}{d\mu'^n} \cdot \left\{ \cos. n\varpi \cdot \cos. n\varpi' + \sin. n\varpi \cdot \sin. n\varpi' \right\};$$

and by combining this with the general term of y', there will result the following expression which is clear of the sines and cosines of variable angles, viz.

$$(1-\mu^2)^{\frac{n}{2}} \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot \left\{ \cos. n\varpi \times \iint \beta^{(n)} \cdot (1-\mu'^2)^n \cdot \frac{d^n C'^{(i)}}{d\mu'^n} \cdot M^{(n)} \cdot d\mu' \cdot d\varpi' + \sin. n\varpi \times \iint \beta^{(n)} \cdot (1-\mu'^2)^n \cdot \frac{d^n C'^{(i)}}{d\mu'^n} \cdot N^{(n)} \cdot d\mu' \cdot d\varpi' \right\}:$$

this expression again comes under the method of No. 4; let the integral $\iint y' \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}$ be denoted, as before, by $2\pi \cdot U^{(i)}$; then the part of $U^{(i)}$ derived from the general term of y', will, by the method alluded to, be thus expressed,

$$(1-\mu^2)^{\frac{n}{2}} \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot \left\{ \cos. n\varpi \times \frac{\int (1-\mu'^2)^i \cdot \frac{d^{i-n} M^{(n)}}{d\mu'^{i-n}} \cdot d\mu'}{2.4.6 \ldots\ldots\ldots\ldots 2i} + \sin. n\varpi \times \frac{\int (1-\mu'^2)^i \cdot \frac{d^{i-n} N^{(n)}}{d\mu'^{i-n}} \cdot d\mu'}{2.4.6 \ldots\ldots\ldots\ldots 2i} \right\}:$$

and if all the parts of $U^{(i)}$ be computed successively by means of this formula, the complete value of that quantity will be found by collecting them all into one sum.

Having thus determined the value of the integral $\iint y' \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}$, denoted by $2\pi \cdot U^{(i)}$, we have

$$B^{(i)} = \alpha \cdot 2\pi \cdot a^{i+3} \cdot U^{(i)}:$$

but it is to be observed, with regard to the case of $i = o$, that

$B^{(0)} = \frac{1}{3}\iint R^3 . d\mu' . d\varpi' = \frac{a^3}{3} . \iint d\mu' . d\varpi' + \alpha . a^3 . \iint y' . d\mu' . d\varpi' . Q^{(0)} = \frac{4\pi}{3} . a^3 + \alpha . 2\pi . a^3 . U^{(0)}$. Therefore the value of V for a point without the surface of the spheroid, will be found by this series, viz.

$$V = \frac{4\pi . a^3}{3r} + \frac{2\pi . \alpha . a^3}{r} \times \left\{ U^{(0)} + \frac{a}{r} . U^{(1)} + \frac{a^2}{r^2} . U^{(2)} + \&c. \right\} . (5)$$

If the attracted point is within the surface, we must operate upon the series investigated in No. 8, of which the general term is,

$$- \frac{r^i}{i-2} . \iint \frac{Q^{(i)} . d\mu' . d\varpi'}{R^{i-2}};$$

and if we substitute $a . (1 + \alpha . y')$ for R, and reject the term which is evanescent as before, and likewise all the terms which are above the first order with regard to α; it will become simply,

$$\frac{\alpha . r^i}{a^{i-2}} . \iint y' . d\mu' . d\varpi' . Q^{(i)} = 2\pi . \alpha a^2 . \frac{r^i}{a^i} . U^{(i)}:$$

with regard to the particular term $\iint \log. R . d\mu' . d\varpi' . Q^{(2)}$, we have only to substitute for log. R, its value log. $a + \alpha . y$; and it will become

$$\alpha . r^2 . \iint y' . d\mu' . d\varpi' . Q^{(2)} = 2\pi . \alpha a^2 . \frac{r^2}{a^2} . U^{(2)};$$

also the term $\frac{1}{2}\iint R^2 . d\mu' . d\varpi' . Q^{(0)}$ will become by substitution,

$$\frac{a^2}{2} . \iint d\mu' . d\varpi' . + \alpha a^2 . \int y' . d\mu' . d\varpi' . Q^{(0)} = 2\pi a^2 + 2\pi . \alpha a^2 . U^{(0)}:$$

these things being observed, the value of V relative to a point within the spheroid, will be expressed by this series, viz.

$$V = -\frac{2\pi r^2}{3} + 2\pi . a^2 + 2\pi . \alpha a^2 . \left\{ U^{(0)} + \frac{r}{a} . U^{(1)} + \frac{r^2}{a^2} . U^{(3)} + \frac{r^3}{a^3} . U^{(4)} + \&c. \right\}. \quad (6)$$

The formulas (5) and (6) enable us to compute the attractions of homogeneous spheroids on a point without or within the surface; and, for a point in the surface, we may make use of either series, observing to put $r = a$ in all the terms multiplied by α, and $r = a . (1 + \alpha . y)$ in the rest. When y' is a finite function, the two expressions for V will both stop. It would be easy to deduce from hence the attractions of heterogeneous spheroids; but having nothing new to offer on this head, I shall refer the reader to LAPLACE's work, No. 14, Chap. 2, Liv. 3e.

The two serieses marked (5) and (6) will be found to be entirely equivalent to the formulas (3)* and (4)† which LAPLACE has given in the second chapter of the third book of the *Mécanique Céleste*: for in effect the coefficient of $\alpha . \frac{a^{i+3}}{r^{i+1}}$, in two of the serieses; and the coefficient of $\alpha . \frac{r^i}{a^{i-2}}$, in the other two, are only different expressions of the same integral $\iint y' . d\mu' . d\varpi' . Q^{(i)}$, the symbol y' being always understood to denote a rational and integral function of three rectangular co-ordinates of a point in the surface of a sphere. In point of result therefore the two methods are one and the same, and the solutions they furnish are both applicable in the same circumstances. Neither of them can be of use, unless the radius of the spheroid be first reduced into such a function as y' is supposed to denote. The one solution can claim no preference

* No. 11. † No. 13.

to the other, except in deducing the same conclusion with greater clearness and expressing it with greater simplicity, and in a form better fitted to fulfil the views of the analyst. In these respects it can hardly be denied that the procedure delivered in the preceding pages has some advantages above that of the author of the *Mécanique Céleste.* The analysis here given is direct; and it exhibits the several coefficients in separate and independent expressions derived immediately from the radius of the spheroid. On the other hand LAPLACE's investigation is indirect; and the coefficients are found successively by decomposing the radius of the spheroid into a series of parts which follow a known law. If we now compare the two methods with respect to the grounds on which the investigations are founded we shall not find the same agreement between them. In this paper it is admitted as a necessary hypothesis, that the radius of the spheroid must be a rational and integral function of three co-ordinates of a point in the surface of a sphere: and, in consequence, the result of the analysis is limited to spheroids of that description. LAPLACE, grounding his investigation on a property which, according to his demonstration, belongs to all spheroids that differ little from spheres, seems to prove that the radius of such a spheroid cannot be an arbitrary expression, and in this inference it is necessarily implied that the radius must be such a function as we have supposed it to be.* What in the one

* Méc. Cél. Liv. 3e, No. 11. In No. 11, by substitution in his fundamental theorem, LAPLACE obtains this formula

$$4\alpha\pi \cdot a^2 y = \frac{U^{(0)}}{a} + \frac{3.U^{(1)}}{a^2} + \frac{5.U^{(2)}}{a^3} + \&c.:$$

of this he remarks, a few lines below; "Cette expression de y n'est donc point

solution is assumed as a necessary hypothesis without which the investigation will not succeed, in the other, is derived as a necessary consequence of a more general supposition. Here then the two methods are so much at variance, that if one be rigorous and exact, the other cannot be exculpated from the charge of erroneous or insufficient reasoning. This contradiction between the preceding analysis and the procedure of LAPLACE is entirely consonant to the conclusions obtained in my former paper alluded to in the beginning of this discourse; and the origin of it is to be sought for in the error I there pointed out in the investigation of that geometer. It cannot be denied that an error of calculation does exist in the demonstration of the theorem on which that author's method is grounded: his reasoning is therefore imperfect and inconclusive; and the inferences he has drawn from it cannot be supported in opposition to a rigorous analysis.

10. The same procedure which has been applied to approximate to the attractions of spheroids differing little from spheres, may likewise be employed to find accurate expressions in serieses of the attractive forces of any spheroid, provided the radius of it be such a function as the analysis requires. In both cases the research turns upon the same sort of integrals. Resume the general term of the series for the attractive force on a point without the surface, viz.

" arbitraire, mais elle derive du developpement en serie, des attractions des sphe-
" roides."

In this formula it is necessarily implied, that y is a rational and integral function of three rectangular co-ordinates of a sphere; because all the terms $\frac{U^{(0)}}{a}$, $\frac{U^{(1)}}{a^2}$ &c. are necessarily such functions.

$$B^{(i)} = \frac{1}{i+3} \cdot \iint R^{i+3} \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)};$$

suppose R to be a function of μ' only, without ϖ'; then, as before,

$$B^{(i)} = \frac{2\pi \,.\, C^{(i)}}{i+3} \cdot \frac{\int (1-\mu'^2)^i \,.\, \frac{d^i \,.\, R^{i+3}}{d\mu'^i} \,.\, d\mu'}{2.4.6 \ldots\ldots\ldots\ldots\ldots 2i}.$$

but if R be a function of the most general kind, then it must be reduced to this form, viz.

$$R^{i+3} = M^{(0)} + (1-\mu'^2)^{\frac{1}{2}} \,.\, M^{(1)} \,.\, \cos.\, \varpi' + \&c.$$
$$+ (1-\mu'^2)^{\frac{1}{2}} \,.\, N^{(1)} \,.\, \sin.\, \varpi' + \&c.:$$

and the several parts that $B^{(i)}$ will consist of must be separately computed, as in the analogous case already considered. The same process will apply when the attracted point is within the surface.

11. To complete the plan of this discourse, it remains that we apply the theory laid down in it to the case of the ellipsoid. Let the semi-axes be k, k', k'', the first being the least of all the three; and let x, y, z, respectively parallel to the same axes, be three co-ordinates of a point in the surface: then will the equation of the solid be

$$\frac{x^2}{k^2} + \frac{y^2}{k'^2} + \frac{z^2}{k''^2} = 1:$$

put $x = R\mu'$; $y = R \,.\, \sqrt{1-\mu'^2} \,.\, \cos.\, \varpi'$; and $z = R \,.\, \sqrt{1-\mu'^2} \,.$ $\sin.\, \varpi'$; then by substitution,

$$R^2 \left\{ \frac{\mu'^2}{k^2} + \frac{(1-\mu'^2)\cos.^2\varpi'}{k'^2} + \frac{(1-\mu'^2)\sin.^2\varpi'}{k''^2} \right\} = 1:$$

farther, let $e = \frac{k^2}{k'^2}$; $f = \frac{k^2}{k''^2}$; and $s = \mu'^2 + e\,(1-\mu'^2)\cos.^2\varpi'$ $+ f \,.\, (1-\mu'^2) \,.\, \sin.^2\varpi'$; then $R = \frac{k}{\sqrt{s}}$: and if this value of R, or the radius of the ellipsoid, be substituted in the general

term of the series for the attractive force on a point within the spheroid (No. 8), that term will become

$$-\frac{1}{i-2}\cdot\frac{r^i}{k^{i-2}}\cdot\iint s^{\frac{i-2}{2}}\cdot d\mu'\cdot d\varpi'\cdot Q^{(i)}.$$

In the first place I say that all the terms in which i is odd are evanescent. For $s=\frac{e+f}{2}+\left(1-\frac{e+f}{2}\right)\cdot\mu'^2+\frac{e-f}{2}\cdot(1-\mu'^2)$ cos. $2\varpi'$; whence it follows that $s^{\frac{i-2}{2}}$ may be expanded into a series of this form, viz. $A^{(0)}+A^{(1)}\cdot\cos.2\varpi'+A^{(2)}\cdot\cos.4\varpi'\ldots A^{(n)}\cdot\cos.2n\varpi'\ldots$ &c.; of which the general term is $A^{(n)}\cdot\cos.2n\varpi'$, and if we combine this quantity with the expansion of $Q^{(i)}$ (No. 5), there will result one term, and only one, independent of sines and cosines, viz.

$$\beta^{(2n)}\cdot(1-\mu^2)^{\frac{2n}{2}}\cdot\frac{d^{2n}C^{(i)}}{d\mu^{2n}}\iint(1-\mu'^2)^{\frac{2n}{2}}\cdot\frac{d^{2n}C'^{(i)}}{d\mu'^{2n}}\cdot A^{(n)}\cdot d\mu'\cdot d\varpi';$$

all the other terms, produced by the multiplication, contain sines or cosines of variable angles; on which account they vanish when they are integrated with regard to $d\varpi'$ between the required limits: since s contains no other power of μ' but μ'^2, it is plain that every coefficient of the developement of $s^{\frac{i-2}{2}}$, as $A^{(n)}$, will be an even function of μ', or will contain only even powers of that quantity: and, because i is odd, therefore $C'^{(i)}$, and all its fluxions of the even orders, will be odd functions of μ': upon the whole then the quantity under the double sign of integration will be an odd function of μ'; or it will be an assemblage of the odd powers of that quantity: therefore the integral, between the limits $\mu'=1$ and $\mu'=-1$, is equal to

nothing (No. 4). Therefore all the terms are evanescent when i is odd.

Again I say that all the terms are evanescent when i is even, except when it is $= 2$. For in this case $s^{\frac{i-2}{2}}$ will be an integer power, and it will contain a finite number of terms which may be generally represented thus, viz.

$$(1-\mu'^2)^{\frac{2n}{2}} \,.\, M^{(n)} \,.\, \cos.\, 2n\varpi' ;$$

$M^{(n)}$ being a rational and integral function of μ': and this quantity when combined with the developement of $Q^{(i)}$, will produce one term, and only one, clear of sines and cosines, viz.

$$(1-\mu^2)^{\frac{2n}{2}} \,.\, \frac{d^{2n}C^{(i)}}{d\mu^{2n}} \,:\!\iint \beta^{(2n)} \,.\, (1-\mu'^2)^{\frac{2n}{2}} \,.\, \frac{d^{2n}C'^{(i)}}{d\mu'^{2n}} \,.\, M^{(n)} \,.\, d\mu' \,.\, d\varpi' :$$

now since μ'^4 is the greatest power in s, the greatest power in $s^{\frac{i-2}{2}}$ will be μ'^{i-2}; therefore $(1-\mu'^2)^{2n} \,.\, M^{(n)}$. cannot contain any power of μ' greater than $i-2$, nor $M^{(n)}$ any greater than $i-2n-2$, which number the dimensions of $M^{(n)}$ cannot pass: but $i-2n$, greater than $i-2n-2$, denotes the dimensions of $\frac{d^{2n}C'^{(i)}}{d\mu'^{2n}}$: therefore, by a property of this sort of integrals already demonstrated (No. 4), the preceding quantity is evanescent. Therefore all those terms of the series are evanescent in which i is an even number; but from this the case of $i = 2$, when the term assumes a particular form, must be excepted.

If now we reject all the terms that have been proved to be evanescent, we shall have, for a point within or in the surface of the ellipsoid,

$$V = -\frac{2\pi . r^2}{3} + \frac{k^2}{2} . \iint \frac{d\mu' . d\varpi'}{s} - \frac{r^2}{2} . \iint \log. s . d\mu' . d\varpi' \, Q^{(2)}:$$

in the last term I have written $-\frac{1}{2}\log. s$ for $\log. \frac{k}{\sqrt{s}} = \log. k - \frac{1}{2}\log. s$; because $\int Q^{(2)} . d\mu' . d\varpi' = o$.

Before we pursue the investigation farther, we shall stop to demonstrate a property of the attractions of a shell of homogeneous matter bounded by the surfaces of two ellipsoids, similar to one another and similarly placed, on a point within the shell. If we suppose k to denote the axis of the greater ellipsoid, and put h for the corresponding axis of the smaller one; then the value of V relatively to the latter solid will be found merely by changing k into h in the last expression; because s contains no quantities but such as are common to the two solids: therefore the value of V, relatively to the shell of matter included between the two surfaces, will be equal to

$$\frac{k^2 - h^2}{2} . \iint \frac{d\mu' \; d\varpi'}{s};$$

a quantity which is independent on the position of the attracted point: therefore the differential coefficients of V for any co-ordinates of the attracted point are evanescent; and consequently so are the attractive forces parallel to the co-ordinates (No. 6). Therefore a material point within such a shell is attracted equally in opposite directions.

Let us now investigate the value of

$$\iint \frac{d\mu' d\varpi'}{s}:$$

put $p = e + (1 - e) . \mu'^2$; $q = f + (1 - f) . \mu'^2$; then $s = p . \cos.^2\varpi' + q . \sin.^2\varpi'$: assume $\sqrt{\frac{q}{p}} . \frac{\sin. \varpi'}{\cos. \varpi'} = \frac{\sin. u}{\cos. u}$; then $\frac{d\varpi'}{s} = \frac{du}{\sqrt{p . q}}$; therefore by restoring the values of p and q, we get

$$\iint \frac{d\mu' \cdot d\varpi'}{s} = \iint \frac{d\mu' \cdot du}{\sqrt{\left\{e+(1-e)\cdot\mu'^2\right\}\cdot\left\{f+(1-f)\cdot\mu'^2\right\}}}:$$

let $\frac{1-e}{e} = \frac{k'^2-k^2}{k^2} = \lambda^2$; $\frac{1-f}{f} = \frac{k''^2-k^2}{k^2} = \lambda'^2$; and

$F = \int \frac{dx}{\sqrt{(1+\lambda^2x^2)\cdot(1+\lambda'^2x^2)}}$ between the limits $x=o$ and $x=1$: then observing that the preceding integrals increase as much from $\mu' = -1$ to $\mu' = 0$, as they do from $\mu' = 0$ to $\mu' = 1$; and likewise that the limits of u are from $u = 0$ to $u = 2\pi$; we shall get

$$\frac{k^2}{2} \cdot \iint \frac{d\mu' \cdot d\varpi'}{s} = \frac{2\pi \cdot k^2}{\sqrt{e \cdot f}} \cdot F.$$

It remains to find the value of

$$\frac{r^2}{2} \cdot \iint \log. s \cdot d\mu' \cdot d\varpi' \cdot Q^{(2)}.$$

Taking the value of $Q^{(2)}$ in terms of γ (No. 3), we have $r^2 Q^{(2)} = r^2 \cdot (\frac{3}{2}\gamma^2 - \frac{1}{2})$: let a, b, c, denote the co-ordinates of the attracted point; then $a = r \cdot \mu$; $b = r \cdot \sqrt{1-\mu^2} \cdot \cos. \varpi$; $c = r \cdot \sqrt{1-\mu^2} \cdot \sin. \varpi$; therefore

$$r \cdot \gamma = a \cdot \mu' + b \cdot \sqrt{1-\mu'^2} \cdot \cos. \varpi' + c \cdot \sqrt{1-\mu'^2} \cdot \sin. \varpi':$$

consequently

$$\begin{aligned} r^2 \cdot Q^{(2)} = {} & a^2 \left(\tfrac{3}{2}\mu'^2 - \tfrac{1}{2}\right) + b^2 \cdot \left\{\tfrac{3}{2}(1-\mu'^2)\cos.^2\varpi' - \tfrac{1}{2}\right\} \\ & + c^2 \cdot \left\{\tfrac{3}{2}(1-\mu'^2)\sin.^2\varpi' - \tfrac{1}{2}\right\} \\ & + 3ab \cdot \mu' \sqrt{1-\mu'^2} \cdot \cos. \varpi' + 3ac \cdot \mu' \sqrt{1-\mu'^2} \cdot \sin. \varpi' \\ & + 3bc \cdot (1-\mu'^2) \cdot \cos. \varpi' \sin. \varpi': \end{aligned}$$

but log. s may be reduced into a series of this form, viz.

$$A^{(0)} + A^{(1)} \cdot \cos. 2\varpi' + A^{(2)} \cdot \cos. 4\varpi' + \&c.$$

and we may neglect all such parts of $Q^{(2)}$ as multiplied by this series would produce only quantities containing sines and

cosines: on this account, we may make

$$r^2 . Q^{(2)} = a^2 . (\tfrac{3}{2}\mu'^2 - \tfrac{1}{2}) + b^2 . \left\{\tfrac{3}{2}(1-\mu'^2)\cos.^2\varpi' - \tfrac{1}{2}\right\}$$
$$+ c^2 . \left\{\tfrac{3}{2}(1-\mu'^2)\sin.^2\varpi' - \tfrac{1}{2}\right\}:$$

therefore,

$$\frac{r^2}{2} . \iint \log. s . d\mu' . d\varpi' . Q^{(2)} = \frac{a^2}{2} . \iint \log. s . d\mu' . d\varpi' . (\tfrac{3}{2}\mu'^2 - \tfrac{1}{2})$$
$$+ \frac{b^2}{2} . \iint \log. s . d\mu' . d\varpi' . \left\{\tfrac{3}{2}(1-\mu'^2)\cos.^2\varpi' - \tfrac{1}{2}\right\}$$
$$+ \frac{c^2}{2} . \iint \log. s . d\mu' . d\varpi' . \left\{\tfrac{3}{2}(1-\mu'^2)\sin.^2\varpi' - \tfrac{1}{2}\right\}.$$

Let the term multiplied by $\frac{a^2}{2}$ be integrated by parts with respect to $d\mu'$, then $\int \log. s . d\mu' . \left(\frac{3\mu'^2}{2} - \frac{1}{2}\right) = \log. s \times \frac{\mu'^3 - \mu'}{2}$ $- \int \frac{\mu'^3 - \mu'}{2} . \frac{\left(\frac{ds}{d\mu'}\right)}{s} . d\mu'$: but $- \frac{\mu'^3 - \mu'}{2} . \left(\frac{ds}{d\mu'}\right) = \mu'^2 - \mu'^2 . s$: therefore, observing that the term without the sign of integration vanishes both when $\mu' = -1$ and $\mu' = 1$; the value of the coefficient of $\frac{a^2}{2}$ will be equal to

$$\iint \frac{\mu'^2 . d\mu' . d\varpi'}{s} - \iint \mu'^2 . d\mu' . d\varpi':$$

and because $\frac{d\varpi'}{s} = \frac{du}{\sqrt{p . q}}$; therefore the first term of the quantity sought will be equal to

$$\pi . a^2 . \int \frac{\mu'^2 . d\mu'}{\sqrt{\left\{e + (1-e) . \mu'^2\right\} . \left\{f + (1-f)\mu'^2\right\}}} - \frac{2\pi . a^2}{3}:$$

which is equal to

$$\frac{2\pi . a^2}{\sqrt{ef}} . \int \frac{x^2 . dx}{\sqrt{(1+\lambda^2 x^2) . (1+\lambda'^2 x^2)}} - \frac{2\pi . a^2}{3},$$

the fluent here being taken from $x = 0$ to $x = 1$. Seeking to express this value by means of the integral F, I have found

in general, $\int \frac{x^2 . dx}{\sqrt{(1+\lambda^2x^2).(1+\lambda'^2x^2)}} = \frac{x^3}{\sqrt{(1+\lambda^2x^2).(1+\lambda'^2x^2)}} + \frac{1}{\lambda} . \left(\frac{dF}{d\lambda}\right) + \frac{1}{\lambda'} . \left(\frac{dF}{d\lambda'}\right)$;

therefore, making $x = 1$, the first term will become,

$$\frac{2\pi . a^2}{\sqrt{ef}} . \left\{ \frac{1}{\sqrt{(1+\lambda^2).(1+\lambda'^2)}} + \frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) + \frac{1}{\lambda'}\left(\frac{dF}{d\lambda'}\right) \right\} - \frac{2\pi . a^2}{3}.$$

With regard to the term containing b^2, it may be changed into an equivalent expression similar to the first term we have just been considering: for if, at entering on this investigation, we had substituted in the equation of the solid, $x = R . \sqrt{1-\mu'^2} . \cos.^2\varpi'$; $y = R' . \mu'$; $z = R . \sqrt{1-\mu'^2} . \sin.^2\varpi'$; which substitutions are entirely arbitrary; we should have found $s = e\mu'^2 + (1-\mu'^2) \cos.^2\varpi' + f \times (1-\mu'^2) \sin.^2\varpi'$; and the term we are seeking, multiplied by b^2, would have been changed into

$$\frac{b^2}{2} . \iint \log. s . d\mu' . d\varpi' . (\tfrac{3}{2}\mu'^2 - \tfrac{1}{2}):$$

and hence by proceeding as before, we derive this value of that term

$$\frac{b^2}{2} . \iint \frac{e . \mu'^2 . d\mu' . d\varpi'}{s} - \frac{b^2}{2} . \iint \mu'^2 . d\mu' . d\varpi':$$

and if we put $p = 1 + (e-1) . \mu'^2$; $q = f + (e-f) . \mu'^2$; also $\sqrt{\frac{q}{p}} . \frac{\sin. \varpi'}{\cos. \varpi'} = \frac{\sin. u}{\cos. u}$: then, $s = p . \cos.^2\varpi' + q . \sin.^2\varpi'$; $\frac{d\varpi'}{s} = \frac{du}{\sqrt{pq}}$; consequently, by substitution, and integrating with regard to u, and confining the integration with regard to μ' between the limits $\mu' = o$ and $\mu' = 1$; we shall get,

$$2\pi . b^2 . \int \frac{e . \mu'^2 . d\mu'}{\sqrt{\left\{1+(e-1).\mu'^2\right\} . \left\{f+(e-f).\mu'^2\right\}}} - \frac{2\pi . b^2}{3}.$$

If we make $\mu' = \frac{x}{\sqrt{e+(1-e)x^2}}$;* the integral in the last expression will be transformed into

$$\frac{1}{\sqrt{ef}} \cdot \int \frac{x^2 \cdot dx^2}{\left\{1+\lambda^2 x^2\right\}^{\frac{3}{2}} \cdot \left\{1+\lambda'^2 x^2\right\}^{\frac{1}{2}}} = -\frac{1}{\sqrt{ef}} \cdot \frac{1}{\lambda} \left(\frac{dF}{d\lambda}\right):$$

therefore the value of this term is

$$-\frac{2\pi \cdot b^2}{\sqrt{ef}} \cdot \frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) - \frac{2\pi \cdot b^2}{3}.$$

And, by proceeding in a manner entirely analogous, it may be shewn that the remaining term multiplied by c^2, is equal to

$$-\frac{2\pi \cdot c^2}{\sqrt{ef}} \cdot \frac{1}{\lambda'}\left(\frac{dF}{d\lambda'}\right) - \frac{2\pi \cdot c^2}{3}.$$

If M denote the mass of the ellipsoid, then $M = \frac{4\pi}{3} \cdot kk'k'' = \frac{4\pi \cdot k^3}{3 \cdot \sqrt{ef}}$; and $\frac{2\pi}{\sqrt{ef}} = \frac{3M}{2k^3}$: therefore by collecting all the parts of V, into one sum, we have

$$\left.\begin{aligned}
V = \frac{3M}{2k} \cdot F - \frac{3M \cdot a^2}{2k^3} \cdot \left\{\frac{1}{\sqrt{(1+\lambda^2) \cdot (1+\lambda'^2)}} + \frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) + \frac{1}{\lambda'}\left(\frac{dF}{d\lambda'}\right)\right\} \\
+ \frac{3M \cdot b^2}{2k^3} \cdot \frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) + \frac{3M \cdot c^2}{2k^3} \cdot \frac{1}{\lambda'}\left(\frac{dF}{d\lambda'}\right). \\
\frac{k'^2 - k^2}{k^2} = \lambda^2; \quad \frac{k''^2 - k^2}{k^2} = \lambda'^2. \\
F = \int \frac{dx}{\sqrt{(1+\lambda^2 x^2) \cdot (1+\lambda'^2 x^2)}} \text{ (from } x = o, \text{ to } x = 1).
\end{aligned}\right\}$$

The case of an oblate elliptical spheroid of revolution corresponds to the supposition of $k' = k''$ or $\lambda = \lambda'$: but in taking the partial fluxions of F we must attend to the peculiarity that takes place when $\lambda = \lambda'$: for in general $dF = \left(\frac{dF}{d\lambda}\right) d\lambda + \left(\frac{dF}{d\lambda'}\right) d\lambda'$; and hence when $\lambda = \lambda'$, $dF = 2\left(\frac{dF}{d\lambda}\right) \cdot d\lambda$: now when $\lambda = \lambda'$, $F = \frac{1}{\lambda}$. arc. tan. λ; consequently $\frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) = \frac{1}{\lambda'}\left(\frac{dF}{d\lambda}\right)$

* Méc. Cél. Liv. 3e, No. 3.

$= -\frac{1}{2\lambda^3}\left(\text{arc. tan. }\lambda - \frac{\lambda}{1+\lambda^2}\right)$: in this case then we shall have

$$V = \frac{3M}{2k} \cdot \frac{1}{\lambda} \cdot \text{arc. tan. }\lambda - \frac{3M \cdot a^2}{2k^3} \cdot \frac{1}{\lambda^3} \cdot \left\{\lambda - \text{arc. tan. }\lambda\right\}$$
$$- \frac{3M}{2k^3} \cdot (b^2 + c^2) \cdot \frac{1}{2\lambda^3} \cdot \left\{\text{arc. tan. }\lambda - \frac{\lambda}{1+\lambda^2}\right\}.$$

If this value of V be substituted in the equation of the surface of a homogeneous fluid mass which is in equilibrium by the joint effect of the attractions of its molecules and a rotatory motion;* it will be proved that the oblate spheroid satisfies the conditions of equilibrium, and the relation between the velocity of rotation and the eccentricity of the spheroid will likewise be determined.

* Méc. Cél. Liv. 3e, No. 23 et 24.

III. *An Account of some Peculiarities in the Structure of the Organ of Hearing in the Balæna Mysticetus of* Linnæus. *By* Everard Home, *Esq. F. R. S.*

Read December 12, 1811.

In the year 1799, I laid before the Society some observations on the Structure of the Membrana Tympani, in consequence of having found that in the elephant that membrane has a muscular structure.

In the elephant the membrana tympani has a greater breadth than in any other animal I have had an opportunity of examining; the structure of its different parts is therefore rendered more conspicuous, which enabled me to discover the muscular fibres.

With a view to prosecute this inquiry, I have ever since that time been desirous of an opportunity of examining the membrana tympani in the balæna mysticetus, having no doubt of its being of such extent as to shew the structure to advantage; and by the kind attention of Mr. Scoresby, Jun. of Whitby, I have now succeeded: that gentleman in his last voyage in the Greenland whale fishery, procured for me the cranium of a cub of the balæna mysticetus whose extreme length was from sixteen to seventeen feet, and its circumference from twelve to thirteen feet. This cranium was put into a cask of salt water, and arrived in London in good condition, the different parts of the organ of hearing being in a state fitted for dissection.

In my examination of the organ of hearing in this young whale, I find there is a peculiarity in its mechanism not met with in the smaller species of whale, and to which there is nothing similar in other animals. As this very singular mechanism is not noticed by either Camper or Monro, and is only glanced at by Hunter, to whom it was imperfectly known, I shall give a description of it, and in doing so, mention the other parts of the organ, so far as will be necessary to make myself understood.

The cranium had been deprived of the external skin, consequently the outward aperture of the ear had been removed; a very small portion, however, of the external meatus could have been cut off, since the dark coloured cuticular lining, which is a continuation of the outward covering of the head, extended a little way into the tube.

The meatus externus was five inches and a half long, and probably an inch had been cut off. Near this part the tube was one quarter of an inch in diameter, in the middle it was narrower, and near the membrana tympani one inch and one-third of an inch.

By comparing the measurement of the length of the tube with that of the skull in this whale, and with the large skull in the Hunterian Museum, the meatus externus in the full grown whale will be found to be about two feet six inches in length.

The membrana tympani is one inch and one-tenth of an inch in diameter, where it is attached to the bone; instead of being concave, as in other animals, towards the meatus externus, it is convex, and projects nearly an inch into that tube. Its external surface is composed of a smooth firm cuticular

covering which readily separates as soon as putrefaction comes on; under this is a strong compact membrane, and when that is removed there is a regular layer of muscular fibres; these go over the whole of the embossed part, having their origin at the edge of the bone to which the membrane is attached, and terminating in the bone on the opposite side. This arrangement of muscular fibres, differs from that of the elephant, where the central part is tendinous. The muscular fibres have a membranous lining between them and the cavity of the tympanum.

From this description of the membrana tympani, it is evident that there is no connexion between it and the ossicula auditûs, or small bones of the ear, which Mr. HUNTER supposed to be the case in consequence of having found that it was so in the porpoise.* He says, " in the piked whale the " membrana tympani is projecting, and returns back into the " meatus externus for above an inch in length, is firm in tex- " ture, with thick coats, is hollow on the inside, and its mouth " communicating with the tympanum, one side being fixed to " the malleus, similar to the tendinous process which goes from " the inside of the membrana tympani in the others."

The fact is, that there is no connexion whatever between the membrana tympani and the malleus, as will be explained; but as that circumstance forms the great peculiarity in the organ of this species of whale, I thought it right to quote what he had stated on this subject.

Having pointed out that there is no direct connexion between the membrana tympani and the ossicula auditûs, as in

* Observations on the Structure and Œconomy of Whales. Phil. Trans. Vol. LXXVII. p. 371.

other animals, and also shewn that Hunter, unwilling to believe that there could be so great a deviation from the ordinary construction of this organ, was led into an error, which I can only attribute to his having formed to himself too strong a chain of analogies, I shall proceed in my description of the organ.

Immediately behind the membrana tympani is a large cavity formed principally by the concave surface of a large hard bone peculiar to the whale, in the substance of which there is more earthy matter than in almost any other bone met with in animal bodies. In its form it is not very unlike the shell called the concha Veneris, to which it has been compared.

The cavity of the tympanum is of an oval shape, one end of which is bounded by the membrana tympani, the other forms the entrance of the Eustachian tube, and there the cavity is surrounded wholly by membrane inclosed in the substance of the skull. The large concave bone is only slightly connected with the petrous portion of the temporal bone, and is imbedded in a fatty substance of nearly an inch in thickness, with a smooth external surface.

The Eustachian tube is two inches and a half long, it opens externally into the canal leading to the blow-hole; its internal surface is honey-combed, which gives it a glandular appearance, and there are chords and septa crossing from side to side in different places: where it opens into the cavity, it has a valvular structure. The cavity, as it corresponds in its principal uses with the tympanum of other animals, although it does not, as in them, contain the ossicula auditûs, deserves to be called by the same name; it is equal in size to a pint measure, and can only be filled from the Eustachian tube, there being

no other opening into it by which it can communicate externally.

Within the cavity of the tympanum, close to the bony rim to which the membrana tympani is attached, there is a membranous fold fixed at one end to the centre of a slight protuberance on the concave surface of the large hollow bone, and stretched across the cavity, its loose upper edge forming a line across the centre of the hollow of the membrana tympani, the other end passing beyond the cavity to be attached to the short handle of the malleus, which is situated immediately behind the membranous lining of the tympanum. The long handle of the malleus is left loose. The incus and stapes have the same relative situation to one another as in the human ear, differing in nothing but being contained in a cavity distinct from that of the tympanum. The appearance of an os orbiculare is wanting.

The other parts of the organ, the vestibulum, semi-circular canals and cochlea, and the meatus internus through which the nerves from the brain pass to be distributed to these parts, do not differ materially from what is met with in the human ear. As the parts which have been described are delineated in the annexed drawings, I have been less minute in my description than I should have otherwise thought it necessary to be.

From the mechanism which has been described, it is evident that the impulses made on the membrana tympani are not immediately communicated to the ossicula auditûs as in other animals; they are only communicated to the tympanum and thence to the chord stretched across its cavity.

The membrana tympani by its muscular structure has the means within itself of adjustment to different sounds, while

the animal is under water; but the degree of pressure to which it is liable, is incompatible with the nicer vibrations required to impress the internal organ so as to convey to it distinct sounds, and it is for this last purpose that the membrane is stretched across the cavity of the tympanum.

This membrane, from being connected with the concave bone, will have its vibrations increased, and the bone being imbedded in a fatty covering, none of the vibratory motion impressed upon it can be carried off from the opposite side, but the whole will be communicated to the malleus, and so on to the cochlea and semi-circular canals.

EXPLANATION OF THE PLATES.

PLATE I.

An external view of the parts in which the organ of hearing is contained.

aa. The external surface of the large concave bone which forms the cavity of the tympanum.

b. The petrous portion of bone, in which the cochlea and semi-circular canal are contained.

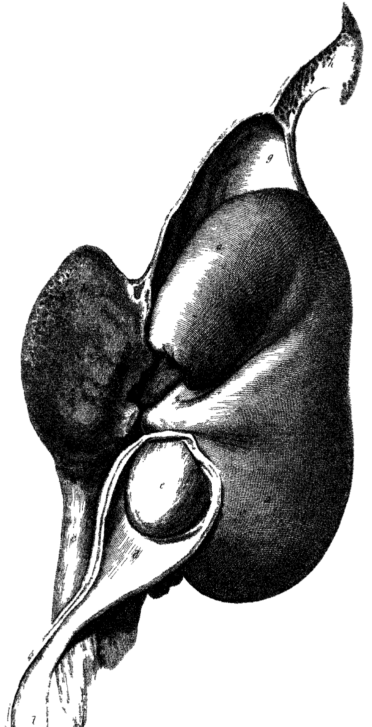
c. A bony process connecting the petrous portion with the skull.

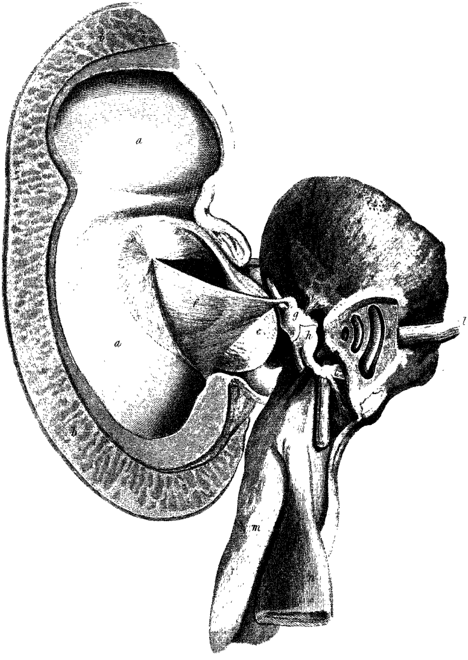
dd. The meatus auditorius externus.

e. The surface of the membrana tympani to show its muscular fibres, the external covering having been removed.

f. A portion of the malleus, one of the small bones of the ear.

g. The anterior portion of the cavity of the tympanum





W. Clift del.

J. Basire sc.

which extends beyond the concave bone, laid open to show the termination of the Eustachian tube.

h. The internal surface of the Eustachian tube.

i. The opening of the Eustachian tube into the nostril.

Plate II.

aa. The internal surface of the concave bone.

bb. The fatty case in which it is inclosed.

c. A convexity covered by a thin ligamentous periosteum, whose fibres are radiated and connect the membrana tympani, as well as the membranous fold *f*, to the bone.

d. The hollow formed on the inside of the membrana tympani.

e. The external surface of the membrana tympani.

f. The membrane stretched across from the concave bone to the malleus.

g. Malleus.

h. Incus.

i. Stapes.

k. Cochlea.

l. Auditory nerve.

m. The bone connecting the petrous portion to the skull.

n. A cartilage which had been cut through in preparing the cranium; the part with which the other extremity was connected has not been ascertained.

IV. *Chemical Researches on the Blood, and some other Animal Fluids. By* William Thomas Brande, *Esq. F.R.S. Communicated to the Society for the Improvement of Animal Chemistry, and by them to the Royal Society.*

Read November 21, 1811.

SECTION I.

Introduction.

In the following pages I shall have the honour of laying before this Society an account of some experiments upon the blood, which were originally undertaken with a view to ascertain the nature of its colouring matter. The difficulties attendant on the analysis of animal substances have rendered some of the results less decisive than I could have wished, but I trust that the general conclusions to which they lead, will be deemed of sufficient importance to occupy the time of this body.

The existence of iron in the blood was first noticed by Menghini,* and its peculiar red colour has been more recently attributed to a combination of that metal with phosphoric acid, by M. M. Fourcroy and Vauquelin.† The

* Vincentius Menghinus de Ferrarum Particularum Progressu in Sanguinem. Comment. Acad. Bonon. T. 2, P. 2, page 475.

† Système des Conn. Chym. Vol. 8, p.

very slight discoloration occasioned by the addition of infusion of galls to a solution of the colouring matter, under circumstances most favourable to the action of that delicate test of iron, first led me to doubt the inferences of those able chemists, and subsequent experiments upon the combinations to which they allude, tended to confirm my suspicion, and induced me to give up no inconsiderable portion of the time which has elapsed since the last meeting of this Society, to the present investigation.

An examination of the chyle and of lymph, in order to compare their composition with that of the blood, formed an important part of this inquiry, especially as those fluids have not hitherto been submitted to any accurate analysis, on account of the difficulty of procuring them in sufficient quantities, and in a state of purity. Whilst engaged in assisting Mr. HOME in his physiological researches, several opportunities occurred of collecting the contents of the thoracic duct under various circumstances, and in different animals; on other occasions Mr. BRODIE has kindly furnished me with the materials for experiment.

SECTION II.

On the Composition of Chyle.

The contents of the thoracic duct are subject to much variation. About four hours after an animal has taken food, provided digestion has not been interrupted, the fluid in the duct may be regarded as pure chyle; it is seen entering by the lacteals in considerable abundance, and is of an uniform white-

ness throughout. At longer periods after a meal, the quantity of chyle begins to diminish, the appearance of the fluid in the duct is similar to that of milk and water; and lastly, where the animal has fasted for twenty-four hours or longer, the thoracic duct contains a transparent fluid which is pure lymph.

A. The chyle has the following properties.

1. When collected without any admixture of blood, it is an opaque fluid of a perfectly white colour, without smell, and having a slightly salt taste, accompanied by a degree of sweetness.

2. The colour of litmus is not affected by it, nor that of paper stained with turmeric, but it slowly changes the blue colour of infusion of violets to green.

3. Its specific gravity is somewhat greater than that of water, but less than that of the blood; this, however, is probably liable to much variation.

4. In about ten minutes after it is removed from the duct, it assumes the appearance of a stiff jelly, which in the course of twenty-four hours gradually separates into two parts, producing a firm and contracted coagulum, surrounded by a transparent colourless fluid. These spontaneous changes, which I have observed in every instance where the chyle was examined at a proper period after taking food, are very similar to the coagulation of the blood and its subsequent separation into serum and crassamentum; they are also retarded and accelerated by similar means.

B. 1. The coagulated portion bears a nearer resemblance to the caseous part of milk than to the fibrine of the blood.

2. It is rapidly dissolved by the caustic and subcarbonated

alkalies. With solutions of potash and soda, it forms pale brown compounds, from which, when recent, a little ammonia is evolved. In liquid ammonia the solution is of a reddish hue.

3. The action of the acids upon these different compounds is attended with nearly similar phenomena, a substance being separated intermediate in its properties between fat and albumen. Nitric acid added in excess redissolves this precipitate in the cold, and sulphuric, muriatic, and acetic acids when boiled upon it for a short time.

4. Neither alcohol nor ether exert any action upon the coagulum of chyle; but of the precipitate from its alkaline solution, they dissolve a small portion, which has the properties of spermaceti: the remainder is coagulated albumen.

5. Sulphuric acid very readily dissolves this coagulum, even when diluted with its weight of water; and with the assistance of heat, it is soluble in a mixture of one part by weight of acid, with four of water; but when the proportion of water is increased to six parts, the dilute acid exerts no action upon it. I was surprised to find that the alkalies produced no precipitation in these sulphuric solutions when heat had been employed in their formation, and where a small proportion only of the coagulum had been dissolved, and was therefore led to exàmine more particularly the changes which the coagulum had undergone by the action of the acid.

On evaporating a solution of one drachm of the coagulum in two ounces of dilute sulphuric acid (consisting of one part by weight of acid with three of water) down to one ounce, a small quantity of carbonaceous matter separated, and the solution had the following properties.

It was transparent, and of a pale brown colour.

Neither the caustic nor carbonated alkalies produced in it any precipitation, when added to exact saturation of the acid, or in excess.

Infusion of galls, and other solutions containing tannin, rendered the acid solution turbid, and produced a more copious precipitation in that which had been neutralized by the addition of alkalies.

When evaporated to dryness, carbonaceous matter was deposited, and sulphurous acid evolved, with the other usual products of these decompositions.

6. On digesting the coagulum in dilute nitric acid, consisting of one part by weight of the acid to fifteen of water, it was speedily rendered of a deep brown colour, but no other apparent change was produced for some weeks, when on removing it from the acid at the end of that period, it had acquired the properties of that modification of fat which is described by FOURCROY under the name of adepocire.*

A mixture of one part of nitric acid with three of water, acted more rapidly upon the coagulum of chyle; a portion of it was dissolved, and when the acid was carefully decanted from the remainder, it was found to possess the properties of gelatine. But when heat was applied, or when a stronger acid was employed, the action became more violent, nitrogen and nitric oxide gas were evolved, and a portion of carbonic acid and of oxalic acid were produced.

7. Muriatic acid in its undiluted state does not dissolve the coagulum of chyle, but when mixed with an equal quantity of water, or even more largely diluted, it dissolves it with facility,

* Mem. de l'Acad. des Sciences, 1789.

forming a straw-coloured solution, which is rendered turbid when the alkalies are added to exact saturation, but no precipitate falls, nor can any be collected by filtration. When either acid or alkali are in excess in this solution, it remains transparent.

8. Acetic acid dissolves a small portion of the coagulum of chyle, when boiled upon it for some hours. As the solution cools, it deposits white flakes, which have the properties of coagulated albumen.

9. The action of oxalic acid is nearly similar to that of the acetic, but neither citric, nor tartaric acid, exert any action upon this coagulum.

10. The destructive distillation of this substance affords water slightly impregnated with carbonate of ammonia, a small quantity of thin fetid oil and carbonic acid and carburetted hydrogen gas.

The coal which remains in the retort is of difficult incineration; it contains a considerable portion of muriat of soda and of phosphat of lime, and yields very slight traces of iron.

C. 1. The serous part of the chyle becomes slightly turbid when heated, and deposits flakes of albumen.

2. If after the separation of this substance the fluid be evaporated to half its original bulk, at a temperature not exceeding 200° FAHRENHEIT; small crystals separate on cooling, which, as far as I have been able to ascertain, bear a strong resemblance to sugar of milk: they require for solution about four parts of boiling water, and from sixteen to twenty parts of water of the temperature of 60°. They are sparingly soluble in boiling alcohol, but again deposited as the solution cools. At common temperatures alcohol exerts no action upon

them. The taste of their aqueous solution is extremely sweet. By nitric acid they are converted into a white powder of very small solubility, and having the properties of saccholactic acid, as described by SCHEELE.*

The form of the crystals I could not accurately ascertain even with the help of considerable magnifiers. In one instance they appeared oblique six-sided prisms, but their terminations were indistinct.

Some of the crystals heated upon a piece of platina in the flame of a spirit lamp, fused, exhaled an odour similar to that of sugar of milk, and burnt away without leaving the smallest perceptible residuum.

3. The destructive distillation of the serous part of chyle afforded a minute quantity of charcoal, with traces of phosphate of lime and of muriate of soda and carbonate of soda.

SECTION III.

Analysis of Lymph.

The fluid found in the thoracic duct of animals that have been kept for twenty-four hours without food, is perfectly transparent and colourless, and seems to differ in no respect from that which is contained in the lymphatic vessels. It may therefore be regarded as pure lymph.

It has the following properties.†

1. It is miscible in every proportion with water.

* Chemical Essays, No. XVII.

† The term lymph has been applied indiscriminately to the tears, to the matter of encysted dropsy, and to some other animal fluids. Vide AIKIN's Dictionary of Chemistry and Mineralogy, Art. Lymph.

2. It produces no change in vegetable colours.

3. It is neither coagulated by heat, nor acids, nor alcohol, but is generally rendered slightly turbid by the last re-agent.

4. When evaporated to dryness, the residuum is very small in quantity, and slightly affects the colour of violet paper, changing it to green.

5. By incineration in a platina crucible the residuum is found to contain a minute portion of muriate of soda; but I could not discover in it the slightest indications of iron.

6. In the examination of this fluid, I availed myself with some advantage of those modes of electro-chemical analysis, which on a former occasion I have described to this Society.*

When the lymph was submitted to the electrical action of a battery, consisting of twenty pairs of four inch plates of copper and zinc, there was an evolution of alkaline matter at the negative surface, and portions of coagulated albumen were separated. As far as the small quantities on which I operated enabled me to ascertain, muriatic acid only was evolved at the positive surface.

SECTION IV.

Some Remarks on the Analysis of the Serum of Blood.

This fluid has been so frequently and fully examined by chemists, that I shall not enter into a detailed account of its composition, but merely state such circumstances respecting it as relate particularly to the present inquiry, and have not hitherto been noticed by the experimentalists to whom I have alluded.

* Phil. Trans. 1809, p. 373

The fluid which oozes from serum that has been coagulated by heat, and which, by physiologists, is termed *serosity*, is usually regarded as consisting of gelatine, with some uncombined soda, and minute portions of saline substances, such as muriate of soda and of potash, and phosphate of lime, and of ammonia. Dr. BOSTOCK regards it as mucus.*

From some experiments which I made upon the serum of blood, on a former occasion, I was induced to regard the serosity as a compound of albumen with excess of alkali, and to consider the coagulation of the serum analogous to that of the white of egg, and of the other varieties of liquid albumen.

To ascertain this point, and to discover whether gelatine exists in the serum, I instituted the following experiments.

Two fluid ounces of pure serum were heated in a water bath until perfectly coagulated: the coagulum, cut into pieces, was digested for some hours in four fluid ounces of distilled water, which was afterwards separated by means of a filter.

The clear liquor reddened turmeric paper, and afforded a copious precipitation on the addition of infusion of galls, and when evaporated to half an ounce, it gelatinised on cooling. It was rendered very slightly turbid by the addition of dilute sulphuric and muriatic acid; but alcohol produced no effect.

From the result of these trials, it might have been concluded that gelatine was taken up by the water, but as an alkaline solution of albumen forms an imperfect jelly when duly concentrated, and as albumen and gelatine are both precipitated by tannin, I was inclined to put little reliance on the appearances just described, until I had examined the solution by the more accurate method of electrical decomposition.

* Transactions of the Medical and Chirurgical Society of London, Vol. I. p. 73.

Upon placing it in the Voltaic circuit my suspicions were justified, by the rapid coagulation which took place in contact with the negative wire. I therefore made some other experiments in order to corroborate this result.

One fluid ounce of pure serum was dissolved in three of distilled water: the conductors from a battery of thirty pairs of four inch plates were immersed in this solution at a distance of two inches from each other; the electrization was continued during three hours and a half, the solid albumen being occasionally removed; at the end of that period, no further coagulation took place, and a mere decomposition of the water was going on.

Having ascertained in previous researches, that gelatine is not altered during the electrical decomposition of its solution carried on as just described, my object in this experiment was, to ascertain whether any gelatine remained after the complete separation of the albumen had been effected. I accordingly examined the water from which the coagulated albumen had been removed, and found that it was not altered by infusion of galls, nor did it afford any gelatine when evaporated to dryness.

Two fluid ounces of dilute muriatic acid were added to one of serum. The mixture immediately assumed a gelatinous appearance; it was heated, and a more perfect coagulation of the albumen took place; the liquid part was separated by a filter. No effect was produced upon it by Voltaic electricity, nor did infusion of galls occasion any precipitation.

I repeated the first experiment with the addition of twenty drops of a solution of isinglass to the serum. The liquid which now separated, after the albumen had been entirely coagulated

by the action of electricity, was copiously precipitated by infusion of galls.

It may be inferred from these experiments, that gelatine does not exist in the serum of the blood, and that the serosity consists of albumen in combination with a large proportion of alkali, which modifies the action of the re-agents commonly employed, but which is readily separated by electrical decomposition.

To ascertain whether iron exists in the serum of the blood, one pint was evaporated to dryness in a crucible, and gradually reduced to a coal, which was incinerated and digested in muriatic acid, to which a few drops of nitric acid were added; some particles of charcoal remained undissolved; the solution was saturated with ammonia, which afforded a copious precipitation of phosphate of lime, accompanied with slight traces only of oxide of iron.

SECTION V.

Some Experiments upon the Coagulum of Blood.

Mr. HATCHETT's valuable researches on the chemical constitution of the varieties of coagulated albumen, have shewn that that substance varies but little in its properties, whether obtained from the crassamentum of the blood, or from washed muscular fibre, or other sources; but that the proportion of earthy and saline matter is different in the different varieties.*

It will also be remarked, on referring to the dissertation which I have just quoted, that the ashes obtained by incinerat-

* Phil. Trans. 1800, p. 384.

ing the coal left after the destructive distillation of albumen, did not contain any appreciable proportion of iron.

Assuming the existence of iron in the colouring matter of the blood, I made the following experiments upon the crassamentum of that fluid.

Two pints of blood were collected in separate vessels. The one portion was allowed to coagulate spontaneously, the other was stirred for half an hour with a piece of wood, so as to collect the coagulum, but to diffuse the principal part of the colouring matter through the serum. These two portions of coagulum were now dried in a water-bath, and equal weights of each reduced in a platina crucible to the state of coal, which afterwards was incinerated. The ashes were digested in dilute nitro-muriatic acid, and the solution saturated with liquid ammonia, in order to precipitate the phosphate of lime as well as any iron which might have been present.

The precipitates were collected, dried, and treated with dilute acetic acid, by which they were almost entirely dissolved, some very minute traces only of red oxide of iron remaining, the quantity of which was similar in both cases, and so small as nearly to have escaped observation.

It is reasonable to infer, that if the colouring matter of the blood were constituted by iron in any state of combination, that a larger relative proportion of that metal would have been discoverable in the former than in the latter coagulum; but frequent repetitions of these experiments have shewn that this is not the case, and the following result appears to complete the evidence on this subject.

The colouring matter of a pint of blood was diffused by agitation through the serum, from which it was allowed

gradually to subside, the coagulum having been removed: after twenty-four hours, the clear serum was decanted off, and the remainder, containing the colouring matter, after having been evaporated to dryness, was incinerated, and the ash examined as in former experiments. But the traces of iron were here as indistinct as in the other instances above mentioned, although a considerable quantity of the colouring matter had been employed.

The minutiæ of analysis I have purposely excluded, as leading into details which would exceed the proper limits of this paper, and unnecessary in the present investigation; I shall now merely dwell on the principal results which have been obtained, and on the general conclusions which these afford.

SECTION VI.

Researches on the colouring Matter of the Blood.

1. To procure this substance for experiments, I generally employed venous blood which had been stirred during its coagulation; the fibrina is thus removed, and the colouring matter diffused through the serum, from which it gradually subsides, being difficultly soluble in that fluid; on decanting off the supernatant serum, the colouring matter remains in a very concentrated form. When other modes of procuring it were employed they will be particularly mentioned; but as I have not found the serum which is retained interfere much with the effects of various agents upon the colouring principle, the method just noticed was commonly adopted.

2. When the colouring matter thus collected is microsco-

pically examined, it seems, as LEWENHOECK first observed,* to consist of minute globules. These are usually described as soluble in water, a circumstance which my own observations led me to doubt, and which the more accurate experiments of Dr. YOUNG, an account of which, intended for publication, he has kindly permitted me to peruse, have completely disproved.

3. The effect of water upon the red globules, is to dissolve their colouring matter, the globule itself remaining colourless, and, according to Dr. YOUNG, floating upon the surface.

This aqueous solution is of a bright red colour, and not very prone to putrefaction. When heated, it remains unaltered at temperatures below 190° or 200° FAHRENHEIT; at higher temperatures it becomes turbid, and deposits a pale brown sediment: if in this state it be poured upon a filter, the water passes through without colour, so that exposure to heat not only destroys the red tint, but renders the colouring matter insoluble in water.

Alcohol and sulphuric ether added to this solution also render it turbid, and when these mixtures were filtrated, a colourless and transparent liquor was obtained.

4. The matter remaining upon the filter was insoluble in water, in alcohol, and in sulphuric ether; but when digested in dilute muriatic or sulphuric acid, a portion was taken up forming a brown solution. I regard this soluble portion as a modification of the colouring matter produced by the operation of heat: the insoluble residuum had the properties of albumen.

* HALLER Elem. Physiolog. Vol. I. p. 51.

5. *Effects of Acids on the colouring Matter.*

A. Muriatic acid poured upon the colouring matter of the blood, renders one portion of it nearly insoluble and of a bright brown colour: another portion is taken up by the acid forming a dark crimson solution when viewed by reflected light; but when examined by transmitted light, it has a greenish hue.

This solution remains transparent, and its colour is unimpaired by long exposure to light, either in contact with the air, or when kept in close vessels. At its boiling temperature the colour is also permanent.

Infusion of galls produces no change in this muriatic solution, nor is its colour affected by carbonated alkalies, even when added in considerable excess.

It is rendered brown red by supersaturation with caustic potash, but not with soda, nor ammonia: these, and especially the latter, rather heighten its colour.

When considerably diluted with water its original colour is much impaired, and the green hue, which it always exhibits by transmitted light, becomes more evident.

In preparing this solution, I frequently employed the coagulum of blood cut into pieces, and digested in equal parts of muriatic acid and water, at a temperature between 150° and 200°. In three or four hours the acid was poured off, and filtrated. The clear solution was in all respects similar to that above described, although before filtration it appears of a dirty brown colour.

I evaporated a portion of this muriatic solution in a water-bath, to dryness; it retained its colour to the last, and left a

transparent pellicle upon the evaporating bason, of a dirty red colour: this, when redissolved in muriatic acid acquired its former tint, but the colour of its aqueous solution was nearer brown than red.

B. Sulphuric acid, diluted with eight or ten parts of water, forms an excellent solvent of the colouring principle of the blood.

It may be employed in a more concentrated state; but the bright colour of the solution is in that case apt to be impaired, and when more largely diluted with water, its action is slow and uncertain. Either the sediment of the colouring matter from the serum, or the crassamentum of the blood, may be indifferently employed in forming these solutions.

When dilute sulphuric acid is added to the colouring matter, it renders it slightly purple; and if no heat be applied, the acid when poured off and filtered, is colourless; so that dilute sulphuric acid when cold, does not dissolve this colouring principle.

One part of the crassamentum of blood cut into pieces, was put into a matrass placed in a sand heat, with about three parts of dilute sulphuric acid. It was kept for twelve hours in a temperature never exceeding 212°, nor below 100°. After twenty-four hours the acid was filtered off, and it exhibited a beautiful bright lilac colour, not very intense, and tainted with green when viewed by transmitted light.

This solution is nearly as permanent as that in the muriatic acid. Some of it which has been kept for a month in an open vessel, often exposed to the direct rays of the sun, is very little altered.

When diluted with two or three times its bulk of water,

the lilac tint disappears, and the mixture is only slightly green.

When exposed to heat, the colour gradually changes as the acid becomes more concentrated by evaporation, and when reduced to about half its bulk the lilac hue is destroyed.

The solutions of pure and carbonated alkalies when added in excess, convert the colour of this sulphuric solution to brownish red; but in smaller quantities, they merely impair it by dilution.

C. Nitric acid, even much diluted, is inimical to the colouring matter of the blood.

A few drops added to the muriatic or sulphuric solutions gradually convert their colour to a bright brown, and larger quantities produce the same change immediately.

The action which this acid exerts upon the colouring matter under other circumstances is nearly similar, and always attended with its decomposition, so that my attempts to procure a red solution in this menstruum uniformly failed of success.

D. Acetic acid dissolves a considerable quantity of the colouring matter of the blood; the solution is of a deep cherry red colour. When somewhat diluted, or when observed in tubes of about a quarter of an inch bore, this solution appears perfectlygreen by transmitted light. In its other habitudes it nearly resembles the muriatic solution.

E. The solution of the colouring matter in oxalic acid is of a brighter red than those hitherto noticed; that in citric acid is very similar to the acetic solution, and with tartaric acid the compound somewhat inclines to scarlet. All these solutions exhibit the green hue, to which I have so often alluded, in a remarkable degree.

6. *Effects of Alkalies on the colouring Principle of the Blood.*

The caustic and the carbonated alkalies form deep red solutions of this substance, which are extremely permanent.

1. Solutions of pure potash, and of the subcarbonate, take up a large proportion of the colouring matter of the blood. The intensity of the colour of this solution, when concentrated, is such, that it appears opaque, unless viewed in small masses, or in a diluted state, when it is of a bright red colour.

2. In soda and its subcarbonate, the solution has more of a crimson hue, which colour is extremely bright in its concentrated state.

3. The solution in liquid ammonia approaches nearer to scarlet than that in which the fixed alkalies are employed.

4. When these alkaline solutions are supersaturated with muriatic acid, or with dilute sulphuric acid, they acquire a colour nearly similar to the original solutions in those acids, which have been above described.

5. Nitric acid added in small quantities, or even to saturation of the alkaline menstruum, heightens the colour of the three compounds; but when there is a slight excess, a tint of orange is produced, which soon passes into bright yellow.

6. The alkaline solutions may be evaporated nearly to dryness without losing their red colours; during the evaporation of the ammoniacal solution, the alkali flies off, and a brown-red solution of the colouring matter in water remains.

Having ascertained the above facts respecting the colouring principle of the blood, I next proceeded to examine how far it was susceptible of entering into those combinations which are peculiar to other varieties of colouring matter.

These experiments I shall detail in the order in which they were made.

1. Some pure alumine was added to a concentrated aqueous solution of the colouring matter of the blood, and after twenty-four hours the mixture, which had been frequently agitated during that period, was poured upon a filter, and the residuum washed with hot distilled water.

The filtrated liquor had lost much of its original colour; the alumine had acquired a red tinge; it was dried at a temperature between 70° and 80°, during which it became brown.

2. Two hundred grains of alum were dissolved in four fluid ounces of a solution of the colouring matter, similar to that employed in the last experiment. The colour of the compound was bright red. Liquid ammonia was added, and the precipitate collected, and carefully dried. It was of a dirty red, and after some days exposure to light, became nearly brown.

From these, and other experiments which I have not thought it necessary to detail, it appears that alumine will not form a permanent red compound with the colouring principle of the blood; I was therefore next induced to employ oxide of tin.

3. Fifty grains of crystallized muriate of tin (prepared by boiling tin filings in muriatic acid, and evaporating the solution), were dissolved in four ounces of the solution of colouring matter, which immediately assumed a purple tint, and became afterwards brown. It was diluted with twice its bulk of water, and put aside in a stopped phial. On examin-

ing it three days afterwards, a small quantity of a bright red powder was observed at the bottom of the phial, which proved to consist of the colouring principle combined with the metallic oxide. A portion of this compound which has been kept in water for some weeks has undergone no change of colour; but when dried by exposure to air, it loses its brilliant tint, and becomes of a dull red hue.

To a compound solution of muriate of tin and colouring matter, similar to that employed in the last experiment, I added a sufficient quantity of solution of potash to decompose the salt of tin. The precipitate thus obtained was collected, and dried by exposure to the air of a warm room. It was of a dull red colour, and has undergone no apparent change by exposure to the joint action of light and air for three weeks.

4. Finding that supertartrate of potash exalted the colour of the blood, I endeavoured to form a compound of it with that substance and oxide of tin, and thus, in some measure, to imitate the process in which cochineal is employed for the production of scarlet dye; but although a bright red compound is produced, when it is dried at a very moderate temperature its colour becomes similar to that of the other combinations which I have described.

These experiments I repeated in various ways, occasionally applying the salt of tin as a mordant to woollen cloth, linen, &c.; but the brilliancy of the colour was never permanent.

5. Having observed that infusion of galls and decoction of oak bark do not impair the colour of the blood, I conceived that solution of tannin might answer the purpose of a mordant, as it is effectually employed by dyers in giving permanence to some of their red colours.

I accordingly impregnated a piece of calico with decoction of oak bark, and afterwards passed it through an aqueous solution of the colouring matter of blood. When dry, it was of a dirty red colour, nearly similar to that which would have been obtained, had no mordant been applied: when however an alkaline solution of the colouring matter was employed, the colour was equal to that of a common madder red, and as far as I have been able to ascertain, it is permanent.

6. A solution of superacetite of lead was impregnated with the colouring matter of the blood. The compound was bright red: no spontaneous change took place in it, and on the addition of an alkali a white precipitate was formed, the fluid retaining its former tint.

From this and other experiments, in which it was attempted to combine oxide of lead with the colouring of the blood, it would appear that there is no attraction between those two substances.

7. The most effectual mordants, which I have discovered for this colouring matter, are some of the solutions of mercury, especially the nitrate, and corrosive sublimate.

Ten grains of nitrate of mercury (prepared with heat and containing the red oxide), were dissolved in two fluid ounces of a solution of the colouring of the blood. After some hours a deep red compound was deposited, consisting chiefly of the metallic oxide combined with the colouring matter, and a small portion of coagulated albumen. The remaining fluid had nearly lost its red colour.

The nitrate of mercury containing the black oxide, produces nearly similar effects, excepting that the colour of the compound is of a lighter red.

When corrosive sublimate is added to the solution of the colouring matter, its tint is instantaneously brightened, and it becomes slightly turbid from the deposition of albumen. If this be immediately separated by a filter, the liquor which passes through gradually deposits a deep red or purplish insoluble precipitate, and if it now be again filtrated the liquid is colourless, the whole of the colouring principle being retained in the compound which remains upon the filter.

By impregnating some pieces of woollen cloth with solution of nitrate of mercury, or of corrosive sublimate, and afterwards steeping them in an aqueous solution of the colouring matter of the blood, I succeeded in giving them a permanent red tinge, unalterable by washing with soap; and by employing the ammoniacal solution of the colouring matter, calico and linen may be dyed with the same mordant.

In these experiments I was much satisfied by the complete separation of the colouring matter from its solutions, which after the process, were perfectly colourless.

SECTION VII.

Some Remarks on the preceding experimental Details.

From the experiments related in the second section of this paper, it appears that sulphuric acid effects changes upon the coagulum of chyle, similar to those which Mr. HATCHETT has observed to result from the action of dilute nitric acid upon the coagulated white of egg. This last substance, however, is not convertible into gelatine by means of sulphuric acid, whereas in these respects the curd of milk resembles that of chyle: this circumstance, as well as the more ready solubility

of the coagulum of chyle in dilute, than in concentrated acids, points out a strong analogy between those two bodies.

The sweet taste of chyle naturally suggested the idea of its containing sugar;* but I am not aware of any direct experiments which have demonstrated its existence, and have therefore detailed minutely such researches as I have been enabled to make upon the subject, hoping at some future period to render them more complete.

The experiments to prove the non-existence of gelatine in the serum of blood, will, I trust, be deemed sufficiently decisive: they shew that that abundant proximate principle of animals is not merely separated from the blood, in which it has been supposed to exist ready formed, but that it is an actual product of secretion.

The proportion of iron afforded by the incineration of several varieties of animal coal, is much less considerable than we have been led to expect, and the experiments noticed in the fifth section, shew that it is not more abundant in the colouring matter of the blood, than in the other substances which were submitted to examination; and that traces of it may be discovered in the chyle which is white, in the serum, and in the washed crassamentum or pure fibrina.

The inferences to which I have alluded, in the first section of this paper, are strongly sanctioned by these facts, and coincide with the opinion which has been laid before the Royal Society, by Dr. WELLS,† respecting the *peculiar nature* of the colouring principle of the blood, and support the arguments which are there adduced.

That the colouring matter of the blood is perfectly independent of iron, is, I conceive, sufficiently evident from its

* FORDYCE on Digestion, 2d Edition, p. 121.

† Phil. Trans. 1797.

general chemical habitudes, and it appears probable that it may prove more useful in the art of dyeing than has hitherto been imagined, since neither the alkalies nor the acids (with the exception of the nitric) have much tendency to alter its hue. The readiness too with which its stains are removed from substances to which no mordant has been applied, seem to render it peculiarly fit for the purposes of the calico-printer. I have not extended these experiments, nor have I had them repeated on a sufficient scale to enable me to draw more general conclusions respecting the possibility of applying them with advantage in the arts: this would have led me into too wide a field, and one not immediately connected with the objects of this Society: the subject, however, appears important.

It is not a little remarkable that blood is used by the Armenian dyers, together with madder, in the preparations of their finest and most durable reds,* and that it has even been found a necessary addition to insure the permanency of the colour.† This fact alone may be regarded as demonstrating the non-existence of iron as the colouring principle of the blood, for the compounds of that metal convert the red madder to gray and black.

Whilst engaged in examining the colouring matter of the blood, I received from Mr. William Money, house surgeon to the general hospital at Northampton, some menstruous discharge, collected from a woman with prolapsus uteri, and consequently perfectly free from admixture of other secretions. It had the properties of a very concentrated solution of the colouring matter of the blood in a diluted serum, and

* Tooke's Russian Empire, Vol. III, p. 497.

† Aikin's Dictionary, Art. Dyeing, and Philos. Magazine, Vol. XVIII.

afforded an excellent opportunity of corroborating the facts respecting this principle, which have been detailed in the preceding pages. Although I could detect no traces of iron, by the usual modes of analysis, minute portions of that metal may, and probably do exist in it, as well as in the other animal fluids which I have examined; but the abundance of colouring matter in this secretion should have afforded a proportional quantity of iron, did any connection exist between them. It has been observed that the artificial solutions of the colouring matter of the blood, invariably exhibit a green tint when viewed by transmitted light: this peculiarity is remarkably distinct in the menstruous discharge.*

I hope that some of the facts furnished by the above experiments, may prove useful to the physiological inquirer: they account for the rapid reproduction of perfect blood after very copious bleedings, which is quite inexplicable upon that hypothesis which regards iron as the colouring matter, and may perhaps lead to the solution of some hitherto unexplained phenomena connected with the function of respiration. There can, I think, be little doubt that the formation of the colouring matter of the blood is connected with the removal of a portion of carbon and hydrogen from that fluid, and that its various tints are dependent upon such modifications of animal matter, and not, as some have assumed, upon the different states of oxidizement of the iron which it has been supposed to contain.

* I could discover no globules in this fluid; and although a very slight degree of putrefaction had commenced in it, yet the globules observed in the blood would not have been destroyed by so trifling a change.

V. *Observations of a Comet, with Remarks on the Construction of its different Parts.* *By* William Herschel, *LL.D. F.R.S.*

Read December 19, 1811.

THE comet which has lately visited the solar system has moved in an orbit very favourably situated for astronomical observations. I have availed myself of this circumstance, and have examined all the parts of it with a scrutinizing attention, by telescopes of every degree of requisite light, distinctness, and power.

The observations I have made have been so numerous, and so often repeated, that I shall only give a selection of such as were made under the most favourable circumstances, and which will serve to ascertain the most interesting particulars relating to the construction of the comet.

As my attention in these observations were every night directed to as many particulars as could be investigated, it will be most convenient to assort together those which belong to the same object; and in the following arrangement I shall begin with the principal part, which is

The planetary Body in the Head of the Comet.

By directing a telescope to that part of the head where with the naked eye I saw a luminous appearance not unlike a star; I found that this spot, which perhaps some astronomers may call a nucleus, was only the head of the comet; but that within

its densest light there was an extremely small bright point, entirely distinct from the surrounding glare. I examined this point with my 20 feet, large 10 feet, common 10 feet, and also with a 7 feet telescope; and with every one of these instruments I ascertained the reality of its existence.

At the very first sight of it, I judged it to be much smaller than the little planetary disk in the head of the comet of the year 1807; but as we are well assured that if any solidity resembling that of the planets be contained in the comet, it must be looked for in this bright point; I have called it the planetary body; in order to distinguish it from what to the naked eye or in small telescopes appeared to be a nucleus, but which in fact was this little body with its surrounding light or head seen together as one object.

With a new 10 feet mirror of extraordinary distinctness, I examined the bright point every fine evening, and found that although its contour was certainly not otherwise than round, I could but very seldom perceive it definedly to be so.

As hitherto I had only used moderate magnifiers from 100 to 160, because they gave a considerable brightness to the point, it occurred to me that higher powers might be required to increase its apparent magnitude; accordingly the 19th of October, having prepared magnifiers of 169, 240, 300, 400, and 600, I viewed the bright point successively with these powers.

With 169 it appeared to be about the size of a globule which in the morning I had seen in the same telescope and with the same magnifier, and which by geometrical calculation subtended an angle of 1″,39.

I suspected that this apparent size of the bright point was

only such as will spuriously arise from every small star-like appearance; and this was fully confirmed when I examined it with 240; for by this its magnitude was not increased; which not only proved that my power was not sufficient to reach the real diameter of the object, but that the light of this point was, like that of small stars, sufficiently intense to bear being much magnified.

I viewed it next with 300, and here again I could perceive no increase of size.

When I examined the point with 400, it appeared to me somewhat larger than with 300; I saw it indeed rather better than with a lower power, and had reason to believe that its real diameter was now within reach of my magnifiers. Curiosity induced me to view it in the 7 feet telescope with a power of 460: and notwithstanding the inferior quantity of light of this instrument, the magnitude was fully sufficient to show that the increase of size in this telescope agreed with that in the 10 feet.

Returning again to the latter I examined the bright point with 600, and saw it now so much better than with 400, that I could keep it steadily in sight while it passed the field of view of the eye-glass.

With this power I compared its appearance to the size of several globules, that have been examined with the same telescope and magnifier, and by estimation I judged it to be visibly smaller than one of 1″,06 in diameter, and rather larger than another of 0″,68.

It should be noticed that I viewed the globules, which were of sealing wax, without sunshine, in the morning after the observation as well as the morning before; referring in one

case the bright point to the globules, and in the other the globules to the bright point.*

The apparent and real Magnitude of the planetary Body.

The size of the bright point being much more like the smallest of the two globules, I shall add one quarter of their difference to 0″,68, and assume the sum, which is 0″,775 as the apparent diameter of the planetary disk.

Then by a calculation from some corrected elements of the comet's orbit, which, though not very accurate, are however sufficiently so for my purpose, I find that the distance of the comet from the earth, at the time of observation, was nearly 114 millions of miles; from which it follows that the bright point, or what we may admit to be the solid or planetary body of the comet, is about 428 miles in diameter.

The Eccentricity and Colour of the planetary Body.

The situation of the bright point was not in the middle of the head, but was more or less eccentric at different times.

The 16th of October that part of the head, which was towards the sun was a little brighter and broader than that towards the tail, so that the planetary disk or point was a little eccentric.

The 17th I found its situation to be a little beyond the centre, reckoning the distance in the direction of a line drawn from the sun through the centre of the head.

The 4th of November it was more eccentric than I had ever seen it before.

* A similar method was used with the comet of 1807. See Phil. Trans. for 1808, page 145.

Nov. 10, I found no alteration in the eccentricity since the last observation.

The colour of the planetary disk was of a pale ruddy tint, like that of such equally small stars as are inclined to red.

The Illumination of the planetary Body.

The smallness of the disk, even when most magnified, rendered any determination of its shape precarious; however had it been otherwise than round, it might probably have been perceived; the phasis of its illumination at the time of observation being to a full disk as 1,6 to 2.

From this as well as from the high magnifying power, which a point so faint could not have borne with advantage, had it shone by reflected light, we may infer that it was visible by rays emitted from its own body.*

The Head of the Comet.

It has already been noticed that the brightest part of the comet seen by the naked eye, appeared to contain a small star-like nucleus. When this was viewed in a night glass, or finder, magnifying only 6 or 8 times, it might still have been mistaken for one; but when I applied a higher power, such as from 60 to 120, it retained no longer this deceptive appearance; which evidently arises from an accumulation of light,

* On the subject of the nature of the light by which we see this comet, I may refer to what has been said in my paper of observations on that of the year 1807. Those who wish also to consult the opinion of an eminent philosopher, whose valuable works on meteorological subjects are well known, will find it expressed at large in a letter from Mr. De Luc, addressed to Mr. Bode, so far back as the year 1799, and reprinted in Mr. Nicholson's Journal, published the 1st of March 1809.

condensed into the small compass of a few minutes; and which of course will vanish when diluted by magnifying.

Sept. 2, I saw the comet at Glasgow, in a 14 feet Newtonian reflector; but being very low, the moon up, and the atmosphere hazy, it appeared only like a very brilliant nebula, gradually brighter in a large place about the middle.

The 9th and 10th of September at Alnwick, I viewed it with a fine achromatic telescope, and found that, when magnified about 65 times, the planetary disk-like appearance seen with the naked eye, was transformed into a bright cometic nebula, in which, with this power, no nucleus could be perceived.

The 18th of September the star-like object in my large 10 feet reflector, when magnified 110 times, had the appearance of a fine globular, lumininous nebula; it seemed to be about 5 or 6 minutes in diameter, of which one or two minutes about the centre were nearly of equal brightness. The small 10 feet showed it in the same manner.

In all my instruments this bright appearance was equally transformed into a brilliant head of the comet, with this difference, that when high powers were applied, the central illumination which moderately magnified, was pretty uniform, became diluted into a gradual decrease from the middle towards the outside; losing itself by imperceptible degrees, especially towards the sides and following parts, into a darkish space, which from observations that will be given hereafter, I take to be a cometic atmosphere.

The Colour and Eccentricity of the Light of the Head.

The colour of the head being very remarkable, I examined it with all my different telescopes; and in every one of them,

its light appeared to be greenish, or bluish green. Its appearance was certainly very peculiar.

The disposition of the light of the head was likewise accompanied with some remarkable circumstances; for notwithstanding a general accumulation about the middle, there seemed to be a greater share of it towards the sun, than a portion in that situation of the circumference was entitled to, had it been uniformly arranged; and if we look upon the head as a coma to the planetary point, the eccentricity of its light will be still more evident; for this point was constantly more or less farther from the sun than the middle of the greatest brightness of the light surrounding it. The eccentricity of the head was indeed so considerable, that considering the difficulty of seeing the point, it might easily have escaped the notice of one who looked for it in the centre of the head.

The apparent and real Magnitudes of the Head.

With an intention to ascertain the dimensions of the various parts of the comet, I viewed the head in the 7, 10, and 20 feet telescopes, and estimated its size by the proportion it bore to the known fields of the eye-glasses that were used. I shall only mention two estimations: September 29, the 10 feet gave its apparent diameter 3′ 0″. With the 20 feet Oct. 6, it was 3′ 45″.

From a calculation of the 20 feet measure, which I prefer, it appears that the real diameter of the head at this time was about 127 thousand miles.

A transparent and elastic Atmosphere about the Head.

In every instrument through which I have examined the comet, I perceived a comparatively very faint or rather darkish interval surrounding the head, wherein the gradually diminishing light of the central brightness was lost. This can only be accounted for by admitting a transparent elastic atmosphere to envelope the head of the comet.

Its transparency I had an opportunity of ascertaining the 18th of September, when I saw three very small stars of different magnitudes within the compass of it; and its elasticity may be inferred from the circular form under which it was always seen. For being surrounded by a certain bright equidistant envelope, we can only account for the equality of the distance by admitting the interval between the envelope and the head of the comet to be filled with an elastic atmospherical fluid.

The Extent of the cometic Atmosphere.

When I examined the comet in the 20 feet telescope the 6th of October, the circular darkish space, which surrounded the brightness, just filled the field of the eye-glass; which gives its apparent diameter 15 minutes. This atmosphere was therefore more than 507 thousand miles in diameter; but its real extent of which, as will be seen, we can have no observation, must far exceed the above calculated dimensions.

The bright Envelope of the cometic Atmosphere.

When I observed the comet at Alnwick in an achromatic refractor with a magnifying power of 65, I perceived that the head of it was partly surrounded by a train of light, which

was kept at some considerable distance by an interval of comparative darkness; and from its concentric figure I call this light an envelope.

The Figure, Colour, and Magnitude of the Envelope.

On viewing this envelope in telescopes that magnify no more than about 16 times, or in finders and night glasses with still lower powers, I found that its shape, as far as it extended, was apparently circular; but that in its course it did not reach quite half way round the head of the comet. A little before it came so far it divided itself into two streams, one passing by each side of the head.

The colour of the envelope in my 7, 10, and 20 feet telescopes had a strong yellowish cast, and formed a striking contrast with the greenish tint of the head.

The distance of the outside of the envelope from the centre of the head, in the direction of a line drawn from it to the sun, was about 9′ 30″; and supposing it to have extended sideways, without increase of distance as far as a semi-circle, this would give its diameter about 19 minutes. By computation therefore its real diameter must have exceeded 643 thousand miles.

The Tail of the Comet.

The most brilliant phenomenon that accompanies a comet is the stream of light which we call the tail. Its length is well known to be variable, but the measures or estimations of its extent cannot be expected to be very consistent from several causes foreign to its actual change.

The 2d of September, the moon being up, the comet very low, and the atmosphere hazy, I could perceive no tail.

The 9th, it had a very conspicuous one, about 9 or 10 degrees in length.

On the 18th, the length was 11 or 12 degrees.

The 6th of October it was 25 degrees.

The 12th I estimated it to be only 17 degrees long.

The 14th it appeared to extend to 17½ degrees.

The 15th, by very careful attention, and in a very clear atmosphere, I found the tail to cover a space of 23½ degrees in length.

The greatest real Length of the Tail.

Of the two observations which were made of the greatest length of the tail of the comet, I prefer that of the 15th of October, on account of the clearness of the night.

The apparent length being 23½ degrees, its real extent, taking into the calculation the oblique position in which we saw it, must have been upwards of 100 millions of miles.

The Breadth of the Tail.

The variations in the breadth of the tail will hardly admit of any description; the scattered light of the sides being generally lost by its faintness in such a manner as to render its termination very doubtful.

The 12th of October its breadth in the broadest part was 6¾ degrees, and about 5 or 6 degrees from the head it began to be a little contracted.

The 15th, it was nearly of the same breadth about the middle of its length.

By calculating from the observation of the 12th, we find that the real breadth of the tail on that day was nearly 15 millions of miles.

The Curvature of the Tail.

The shape of the tail with respect to its curvature is generally considered only as it relates to the direction of the motion of the comet; it is nevertheless subject to variations arising from causes that will be noticed in the next article, but which are not taken into the account of the following observations.

The 9th and 10th of September the curvature of the tail was very considerable.

The 18th, I remarked, that towards the end of the tail its curvature had the appearance as if, with respect to the motion of the comet, that part of the tail were left a little behind the head.

The 17th of October the tail appeared to be more curved than it had been at any time before.

Dec. 2, the flexure of the curvature of the tail, contrary to its former direction, was convex on the following side.

The general Appearance of the Tail.

On account of the great length and breadth of the tail of the comet, a night glass with a large field of view is the most proper instrument for examining its appearance. Mine takes in $4^\circ\ 41'$.

By viewing the comet with this glass I found the tail to be inclosed at the sides by the two streams which I have described as the continuation of the bright arch, or envelope surrounding the head.

Sept. 18, I observed that the two streams or branches arising from the sides of the head scattered a considerable portion of their light as they proceeded towards the end of the tail,

and were at last so much diluted that the whole of the farthest part of the tail, contained only scattered light.

Oct. 12, I remarked that the two streams remained sufficiently condensed in their diverging course to be distinguished for a length of about six degrees, after which their scattered light began to be pretty equally spread over the tail.

Oct. 15. The preceding branch of the tail was 7° 1′ in length. The following was only 4° 41′; which caused the appearance of an irregular curvature.

Nov. 3. The two branches were nearly of an equal length.

Nov. 5. The length of the preceding stream was 5° 16′; that of the following about 4° 41′.

Nov. 9. The two branches might still be seen to extend full 4 degrees, but their light was much scattered.

Nov. 10. The preceding branch was 5° 16′ long; the following one only 3° 31′; the preceding one was also fuller and broader.

In the course of these observations I attended also to the appearance of the nebulosity of the tail.

Sept. 18. The appearance of the nebulosity, examined with a 10 feet reflector, perfectly resembled the milky nebulosity of the nebula in the constellation of Orion, in places where the brightness of the one was equal to that of the other.

Nov. 9. The tail of the comet being very near the milky-way, the appearance of the one compared to that of the other, in places where no stars can be seen in the milky-way, was perfectly alike.

The Return of the Comet to the nebulous Appearance.

From the observations of the decreasing length of the tail, the diminution of brightness and increased scattering of the streams, and from the gradually fainter appearance of the transparent atmosphere, brought on by the contraction and more scattered condition of the envelope, I had reason to suppose that all the still visible cometic phenomena of planetary body, head, atmosphere envelope, and tail, would soon be reduced to the semblance of a common globular nebular; not from the increase of the distance of the comet, which could only occasion an alteration in the apparent magnitude of the several parts, but by the actual physical changes which I observed in the construction of the comet.

The gradual vanishing of the planetary Body.

Nov. 4. 10 feet reflector. I saw the planetary disk with 289. It was rather more eccentric than usual.

Nov. 9, I saw it imperfectly with 169. It was more visible with 240; but the nebulosity of the envelope overpowered its light already so much that no good observations could be made of it.

Nov. 10. Large 10 feet. I had a glimpse of the disk and its eccentricity,

Nov. 13, I tried all magnifiers, but could no longer perceive the planetary body.

The Disappearance of the transparent Part of the Atmosphere under the Cover of the scattered Light of the contracted Envelope.

Nov. 4. In the night-glass, that part of the atmosphere which used to separate the envelope from the head, could no longer be distinguished.

In the 10 feet reflector, with a large double eye-glass, I found the envelope drawn nearer to the head, its central distance at the vertex being less than 7′ 10″; and the atmosphere was almost involved in the scattered haziness of the streams.

Nov. 5. The envelope was still disengaged from the head, but much scattered light had nearly effaced the cometic atmosphere on the side towards the sun.

Nov. 9. The atmosphere was nearly covered by the approximation, or scattering light, of the envelope. Its vertical distance was 5′ 45″.

Nov. 10. The envelope could only be distinguished from the head by a small remaining darkish space, in which the atmosphere might still be seen. The vertical distance of the envelope was 4′ 46″.

Nov. 13. The atmosphere was almost effaced by scattered light towards the sun, but on the opposite side it was darker, or rather more transparent.

Nov. 14, 15, and 16. The atmosphere was gradually more covered in.

Nov. 19. I found in the 10 feet telescope, the envelope so broad and scattered as to leave no room for seeing the atmosphere; and the comet seemed to be fast returning to the mere appearance of a nebula.

Nov. 24. The envelope was turned into haziness; and on the side towards the sun, the comet had already the appearance of a globular nebula, with a faint hazy border.

Dec. 2. The haziness of the border was of a different colour from the light of the head, which preserved its former greenish appearance.

Dec. 9. The envelope, which had been turned into a hazy border of light, in which state I saw it again the 5th, was very unexpectedly renewed. It was however very narrow and much fainter than it used to be. By four measures I found its distance from the centre of the head to be about $4\frac{3}{4}$ minutes.

Dec. 14. The narrow faint envelope of the 9th existed no longer.

If the scattered light near the head should not be raised again, all observations of the atmosphere must be at an end; for the space beyond this light being equally clear, we have nothing left to point out any extent that might be supposed to contain a transparent elastic fluid, notwithstanding it should remain in its former situation.

Uncommon Appearances in the Dissolution of the Envelope.

Nov. 4. 10 feet. The envelope was double towards the sun, and divided itself at each side into three streams; the outside ones being very faint, and of no great length.

Nov. 5. On the preceding side the envelope was very faintly accompanied by an outer one, but not on the following side.

Nov. 13. On the following side the envelope diverged into three streams, the two outside ones being very faint and narrow; but on the preceding side there was but one additional

streamlet, which was at the distance of the outermost one of the opposite side.

Nov. 14. On the preceding side there was a very faint outward stream, and on the following side there was a still fainter and shorter stream, also on the outside.

Dec. 14. There was only one short and faint outside stream at the preceding side.

Uncommon Variations in the Length of the Streams.

It has already been mentioned, that the streams or branches were subject to a considerable difference in their respective lengths; in order if possible to discover the cause of the observed changes, I continued my observations of them.

Oct. 15 and Nov. 5 and 10, the preceding branch was the longest.

The 3d and 9th of November the branches were of equal length.

The 13th, the following was 4° 6′ long, the preceding only 3° 31′.

The 14th. They were both of the equal length of about 3° 31′.

The 15th. The preceding branch was 3° 31′ long, the following 4° 6′.

The 16th. The preceding was 3° 13′ long, the following 3° 48.

The 19th. The branches were equal, and about 4° 23′ long.

Dec. 2. The branches were nearly equal and about 3° 12′ long; they joined more to the sides than the vertex, and had lost their former vivid appearance; their colour being changed into that of scattered light.

The 9th and 14th. The branches were already so much scattered that observations of them could no longer be made with any accuracy.

Alterations in the Angle of the Direction of the Envelope.

Nov. 4. 10 feet reflector. Large double eye-glass. The streams departed from their source in a greater angle of divergence. This probably arose from a contraction of the envelope towards the sun, but not about the root of the streams, where it remained extended as before.

Nov. 13. 10 feet. The angle of the bending of the envelope at its vertex was considerably enlarged. In the night-glass the divergence of the streams themselves was certainly not increased.

Nov. 24. 10 feet. The divergence of the light, which may still be called the envelope, although no longer to be distinguished from the head, was from 60 to 65 degrees; but in the night-glass, the branches which were hardly to be seen were closer together than formerly.

The additional faint duplicates of the envelope Nov. 4, 5, 13, and 14 always departed from the vertex in an angle considerably greater than the permanent interior streams.

The Shortening of the Tail.

The 5th of November, the air being very clear, I found, when attending to the tail of the comet, that its length was much reduced; its utmost extent not exceeding 12½ degrees.

The 9th, it was 10 degrees long.

The 15th. In the night glass the tail was much shortened.

The 16th. With the naked eye the tail was nearly $7\frac{1}{2}$ degrees long.

The 19th. Its length was about 6° 10′.

Dec. 2. The tail was hardly 5 degrees long and of a very feeble light.

The 9th, the length of the tail was not materially altered.

The 14th, it still remained as before, but the end of it was much fainter.

Increasing Darkness between the Streams that inclose the Tail.

The 4th of November the darkness near the head on the side from the sun was grown more conspicuous, and much less filled up with scattered light.

The 5th, the darkness of the atmosphere on the side opposite the sun was stronger than on the sun side.

The 10th. A considerable darkness prevailed between the two branches of the tail.

The 14th. In the tail, close to the head, there was a large space almost free from scattered light; where the small stars of the milky-way are as bright as if nothing had intercepted their light.

The 16th. The space between the streams was of a considerable darkness.

The 19th. 10 feet reflector. The darkness between the streams was increased.

Dec. 9. The space close to the head on the side from the sun was quite dark, or rather transparent.

The 14th. Many small stars of the milky-way were in the dark interval of the tail close to the head of the comet.

Of the real Construction of the Comet, and its various Parts.

Hitherto I have only related the appearances of the several parts of the comet, in order to determine their linear extent; but the observations which are now before us, contain facts that will allow me also to ascertain the construction of the comet and its various parts in their solid dimensions.

From the laws of gravitation we might be allowed to conclude that the planetary body containing the solid matter of the comet must be spherical; but actual observation will furnish a more substantial argument; for in no part of the long, geocentric path described by the comet, did I see its little disk otherwise than round; whereas it would not have preserved this appearance, if its construction were not spherical.

If what has been said in my last paper, when treating of round nebulæ, be remembered, the head of the present comet, which by observation appeared round like a nebula, cannot be supposed to be of any other than a spherical construction. With my collection of round nebulæ the arguments, however, which proved their globular form, rested only, though very soundly, upon the doctrine of chances, and the known effects of gravitation; but here, on the contrary, while nebulæ remain in their places, the geocentric position of the head of the comet has undergone a change amounting to a whole quadrant; in all which time I have observed it to retain its roundness without any visible alteration; from which it necessarily follows that its form is globular.

With regard to its transparent cometic atmosphere, we have not only the constant observations of its roundness, during the abovementioned long period of the comet's motion, to

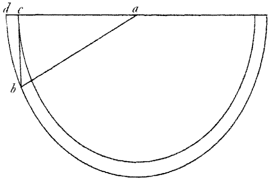
prove it to be spherical; but in addition to this, I have already shown that it is of an elastic nature, for which reason alone, had we no other, its globular figure could not be doubted.

A most singular circumstance, which however must certainly be admitted, is, that the constant appearance of the bright envelope, with its two opposite diverging branches, can arise from no other figure than that of an inverted hollow cone, terminating at its vertex in an equally hollow cap, of nearly a hemispherical construction; nor can the sides or caps of this hollow cone be of any considerable thickness.

The proof of this assigned construction is, that the bright envelope has constantly been seen in my observation as being every where nearly equidistant from the transparent atmosphere; now if that part of it which in a semi-circular form surrounds the comet, on the side exposed to the sun, were not hemispherical, but had the shape of a certain portion of a ring, like that which we see about the planet Saturn, it must have been gradually transformed from the appearance of a semicircle into that of a straight line, during the time that we have seen it in all the various aspects presented to us by a geocentric motion of the comet, amounting to 90 degrees.

That this hemispherical cap is comparatively thin, is proved from the darkness and transparency of that part of the atmosphere which it covers; for had the curtain of light, which was drawn over it, been of any great thickness, the scattered rays of its lustre would have taken away the appearance of this darkness; nor would the atmosphere have remained sufficiently transparent for us to see extremely small stars through it.

It remains now only to account for the semi-circular



Jas. Basire sc.

appearance of the bright envelope; but this, it will be seen, is the immediate consequence of the great depth of light near the circumference, contrasted with its comparative thinness towards the centre. The 6th of October, for instance, the radius of the envelope was 9′ 30″ on the outside, and 7′ 30″ on the inside; and as the greatest brightness was rather nearer to the outside, we may suppose its radius to have been about 8′$\frac{3}{4}$. Then if we compute the depth of the luminous matter at this distance from the centre, we find that it could not be less than 248 thousand miles; whereas in the place where the atmosphere was darkest, its thickness would be only about 50 thousand; so that a superior intensity of light in the ratio of about 5 to 1, could not fail to produce the remarkable appearance of a bright semi-circle, enveloping the head of the comet at the distance at which it was observed.*

I have entered so fully into the formation of the envelope, as the argument, by which its construction has been analysed, will completely explain the appearance of the streams of light inclosing the tail of the comet, and indeed its whole construction.

The luminous matter as it arises from the envelope, of which it is a continuation, is thrown a little outwards, and assumes the appearance of two diverging bright streams or branches; but if the source from which they rise be the circular rim of

* From the measure of the envelope, whose diameter the 6th of October was 643032 miles, we have the radius *a b*, Plate III, 321516. Then if *c d* be 25000, we find the angle *b a c*, of which *a c* is cosine 22° 44′ 37″; and the sine *b c*, which is the depth, will be to the versed sine *c d*, which is the thickness, as 4,972 to 1. And if *a d* is 9′ 30″, the greatest brightness which is at *c* will give the distance *a c* equal to 8′ 45″,7. This calculation being made for that part which is convex towards us, the addition of the concave opposite side will double the dimensions of the depth and thickness.

an hemispherical hollow shell, the luminous matter in its diverging progress upwards can only form a hollow cone; and the appearance of the two bright streams inclosing the tail, after what has been said of the envelope, will want no farther explanation.

Add to this that, having actually seen these brilliant streams remain at the borders of the tail in the same diverging situation during a motion of the comet through more than 130 degrees, the hollow conical form of the comet's tail is in fact established by observation.

The feebler light of the tail between its branches is sufficiently accounted for by the thinness of the luminous matter of the hollow cone through which we look towards the middle of the tail compared with its great depth about the sides; and indeed the comparative darkness of the inside of the cone and transparency of the atmosphere seen through the envelope, bear witness to their hollow construction; for, were these parts solid, both the cone and the hemispherical termination of it must have been much brighter in the middle than towards the circumference, which is contrary to observation.

Of the solar Agency in the Production of Cometic Phenomena.

As we are now in a great measure acquainted with the physical construction of the different parts of the present comet, and have seen many successive alterations that have happened in their arrangement, it may possibly be within our reach to assign the probable manner in which the action of such agents as we are acquainted with has produced the phenomena we have observed.

In its approach to a perihelion, a comet becomes exposed to the action of the solar rays, which, we know, are capable of producing light, heat, and chemical effects. That their influence on the present comet has caused an expansion, and decomposition of the cometic matter, we have experienced in the growing condition of the tail and shining quality of its light, which seems to be of a phosphoric nature. The way by which these effects have been produced may be supposed to be as follows.

The matter contained in the head of the comet would be dilated by the action of the sun, but chiefly in that hemisphere of it which is immediately exposed to the solar influence; and being more increased in this direction than on the opposite side, it would become eccentric, when referred to the situation of the body of the comet; but as the head is what draws our greatest attention, on account of its brightness, the little planetary body would appear to be in the eccentric situation in which we have seen it.

Now, as from observed phenomena, we have good reason to believe the comet to be surrounded by a very extensive, transparent, elastic atmosphere; the nebulous matter, which probably, when the comet is at a distance from the perihelion, is gathered about the head in a spherical form, would on its approach to the sun be greatly rarefied, and rise in the cometic atmosphere till it came to a certain level, where it could remain suspended, for some time, exposed to the continued action of the sun.

In this situation we have had an opportunity of seeing the transparent atmosphere, which, but for the suspension of the nebulous matter, we might never have discovered; and in-

deed; how far it may extend beyond the region which contained the shining substance, we can have no observation to ascertain, on account of its transparency. In consequence of the darkish interval, occasioned by the atmospheric space, the suspended light appeared to us in the shape of a very bright envelope.

The brilliancy of the envelope, and its yellowish colour, so different from that of the head, and probably acquired by its mixture with the atmospheric fluid, are proofs of the continued action of the sun upon the luminous matter, already in so high a state of rarefaction; and if we suppose the attenuation and decomposition of this matter to be carried on till its particles are sufficiently minute to receive a slow motion from the impulse of the solar beams, then will they gradually recede from the hemisphere exposed to the sun, and ascend in a very moderately diverging direction towards the regions of the fixed stars.

That some such operation must have been carried on, is pretty evident from our having seen the gradual rise, and increased expansion of the tail of the comet; and if we saw the shining matter, while suspended in the cometic atmosphere, in the shape of an envelope, it follows that, in its rising condition, it would assume the appearance of those two luminous branches which we have so long observed to inclose the tail of the comet.

The seemingly circular form, and the stream-like appearance of the luminous matter having been already explained, we may now see the reason why it can rise in no other form than the conical; for a whole hemisphere of it being exposed to the action of the sun, it must of course ascend equally every where all around it.

That the luminous matter ascending in the hollow cone, received no addition to its quantity from any other source than the exposed hemisphere, we may conclude from its appearance; which notwithstanding the great circumference of the cone it filled, at the altitude of 6 degrees from the head, was never seen with increased lustre; although the diameter of an annular section of it, in that place, must have been nearly 15 millions of miles, and was but little more than half a million at its rising from the envelope.

This consideration points out the extreme degree of rarefaction of the luminous matter about the end of the tail; for its expansion, while still much confined in the streams, at the altitude which has been mentioned, must have exceeded the density it had at rising about 524 times; but when afterwards it extended itself so as to produce nearly an evenly scattered light over the whole compass of the end of the tail, we may easily conceive to what an extreme degree of rareness its expansion must have been carried.

The vacancy occasioned by the escape of the nebulous matter, which after rarefaction passed from the hemisphere exposed to the sun into the regions of the tail, was probably filled up, either by a succession of it from the opposite hemisphere, or by a rotation of the comet about an axis; and the gradual decomposition of this matter would therefore be carried on as long as any remained to replace the deficiency.

That such a kind of process took place, seems to be supported by the observations which were made during the regression of the comet from its perihelion. For the space between the branches of the tail, very near the head of the comet, became gradually of a darker appearance than before;

which indicated the absence of the nebulous matter that had formerly been lodged there.

A rotatory motion of the comet, which has been suggested, would also explain the frequent variations in the length of the opposite branches which inclosed the tail; for if any portion of the cometary matter should be more susceptible of being thrown into a luminous decomposition than some others, a rotatory motion would bring such more susceptible matter into different situations, and cause a more or less copious emission of it in different places.

The additional short and faint double streams of nebulous light which issued from the vertex or side of the enfeebled envelope, in the gradual regress of the comet, tend likewise to add probability to the conception of a rotatory motion; for the changeable appearance of the situation of these streamlets might arise from a periodical exposition of some remaining small portions of less rarefied matter, when nearly the whole of it had been exhausted.

Of the Result of a Comet's Perihelion Passage.

After having given a detail of phenomena, and entered into a research of the most likely manner in which they were produced, I shall only mention what appears to me to be the most probable consequence of the perihelion passage of a comet.

The quality of giving out light, although it may always reside in a comet, as it does in the immensity of the nebulous matter, which I have shown to exist in the heavens, is exceedingly increased by its approach to the sun. Of this we should not be so sensible, if it were not accompanied with an almost

inconceivable expansion and rarefaction of the luminous substance of the comet about the time of its perihelion passage.

It is admitted, on all hands, that the act of shining denotes a decomposition in which at least light is given out; but that many other elastic volatile substances may escape at the same time, especially in so high a degree of rarefaction, is far from improbable.

Then, since light certainly, and very likely other subtile fluids also escape in great abundance during a considerable time before and after a comet's nearest approach to the sun, I look upon a perihelion passage in some degree as an act of consolidation.

If this idea should be admitted, we may draw some interesting conclusions from it. Let us, for example, compare the phenomena that accompanied the comet of 1807 with those of the present one. The first of these in its approach to the sun came within 61 millions of miles of it; and its tail, when longest, covered an extent of 9 millions. The present one in its perihelion did not come so near the sun by nearly 36 millions of miles, and nevertheless acquired a tail 91 millions longer than that of the former. The difference in their distances from the earth when these measures were taken was but about 2 millions.

Then may we not conclude, that the consolidation of the comet of 1807, when it came to the perihelion, had already been carried to a much higher degree than that of the present one, by some former approach to our sun, or to other similarly constructed celestial bodies, such as we have reason to believe the fixed stars to be?

And that comets may pass round other suns than ours, is

rendered probable from our knowing as yet, with certainty, the return of only one comet among the great number that have been observed.

Since then, from what has been said, it is proved that the influence of the sun upon our present comet has been beyond all comparison greater than it was upon that of 1807; and since we cannot suppose our sun to have altered so much in its radiance as to be the cause of the difference; have we not reason to suppose that the matter of the present comet has either very seldom, or never before passed through some perihelion by which it could have been so much condensed as the preceding comet? Hence may we not surmise that the comet of 1807 was more advanced in maturity than the present one; that is to say, that it was comparatively a much older comet.

Should the idea of age be rejected, we may indeed have recourse to another supposition, namely, that the present comet, since the time of some former perihelion passage, may have acquired an additional quantity (if I may so call it) of *unperihelioned* matter, by moving in a parabolical direction through the immensity of space, and passing through extensive strata of nebulosity; and that a small comet, having already some solidity in its nucleus, should carry off a portion of such matter, cannot be improbable. Nay, from the complete resemblance of many comets to a number of nebulæ I have seen, I think it not unlikely that the matter they contain is originally nebulous. It may therefore possibly happen that some of the nebulæ, in which this matter is already in a high state of condensation, may be drawn towards the nearest celestial body of the nature of our sun; and after their first perihelion pas-

sage round it proceed, in a parabolic direction, towards some other similar body; and passing successively from one to another, may come into the regions of our sun, where at last we perceive them transformed into comets.

The brilliant appearance of our small comet may therefore be ascribed either to its having but lately emerged from a nebulous condition, or to having carried off some of the nebulous matter, situated in the far extended branch of its parabolic motion. The first of these cases will lead us to conceive how planetary bodies may begin to have an existence; and the second, how they may increase and, as it were, grow up to maturity. For if the accession of fresh nebulous matter can be admitted to happen once, what hinders us from believing a repetition of it probable? and in the case of parabolic motions, the passage of a comet through immense regions of such matter is unavoidable.

WM. HERSCHEL.

Slough, near Windsor,
Dec. 16, 1811

VI. *On a gaseous Compound of carbonic Oxide and Chlorine. By* John Davy, *Esq. Communicated by Sir* Humphry Davy, *Knt. LL. D. Sec. R. S.*

Read February 6, 1812.

SINCE the influence of electricity and solar light, as chemical agents, are analogous in many respects, and as the former produces no change in a mixture of carbonic oxide and chlorine, it was natural to infer the same respecting the latter. M. M. GAY LUSSAC and THENARD assert that this is the case; they say that they have exposed a mixture of carbonic oxide and chlorine, under all circumstances, to light, without observing any alteration to take place:* Mr. MURRAY has made a similar statement.†

Having been led to repeat this experiment, from some objections made by the last mentioned gentleman to the theory of my brother, Sir HUMPHRY DAVY, concerning chlorine, I was surprised at witnessing a different result.

The mixture exposed, consisted of about equal volumes of chlorine and carbonic oxide; the gasses had been previously dried over mercury by the action of fused muriate of lime, and the exhausted glass globe into which they were introduced from a receiver with suitable stopcocks, was carefully dried. After exposure for about a quarter of an hour to bright

* Recherches Physico-Chimiques, Tom. II. p. 150.

† NICHOLSON's Journal, Vol. XXX. p. 227.

sunshine, the colour of the chlorine had entirely disappeared; the stopcock belonging to the globe, being turned in mercury recently boiled, a considerable absorption took place, just equal to one-half the volume of the mixture, and the residual gas possessed properties perfectly distinct from those belonging either to carbonic oxide or chlorine.

Thrown into the atmosphere, it did not fume. Its odour was different from that of chlorine, something like that which one might imagine would result from the smell of chlorine combined with that of ammonia, yet more intolerable and suffocating than chlorine itself, and affecting the eyes in a peculiar manner, producing a rapid flow of tears and occasioning painful sensations.

Its chemical properties were not less decidedly marked, than its physical ones.

Thrown into a tube full of mercury containing a slip of dry litmus paper, it immediately rendered the paper red.

Mixed with ammoniacal gas, a rapid condensation took place, a white salt was formed, and much heat was produced.

The compound of this gas and ammonia was a perfect neutral salt, neither changing the colour of turmeric or litmus; it had no perceptible odour, but a pungent saline taste; it was deliquescent, and of course very soluble in water; it was decomposed by the sulphuric, nitric, and phosphoric acids, and also by liquid muriatic acid; but it sublimed unaltered in the muriatic, carbonic, and sulphureous acid gasses, and dissolved without effervescing in acetic acid. The products of its decomposition collected over mercury were found to be the carbonic and muriatic acid gasses; and in the experiment with concentrated sulphuric acid when accurate results could be

obtained, these two gasses were in such proportions, that the volume of the latter was double that of the former.

I have ascertained by repeated trials, both synthetical and analytical, that the gas condenses four times its volume of the volatile alkali, and I have not been able to combine it with a smaller proportion.

Tin fused in the gas in a bent glass tube over mercury, by means of a spirit lamp, rapidly decomposed it; the liquor of Libavius was formed; and when the vessel had cooled, there was not the least change of the volume of the gas perceptible; but the gas had entirely lost its offensive odour, and was merely carbonic oxide; for like carbonic oxide it burnt with a blue flame, afforded carbonic acid by its combustion, and was not absorbable by water.

The effects of zinc, antimony, and arsenic heated in the gas, were similar to those of tin; compounds of these metals and chlorine were formed, and carbonic oxide in each experiment was liberated equal in volume to the gas decomposed. In each instance the action of the metal was quick; the decomposition being completed in less than ten minutes; but though the action was rapid, it was likewise tranquil, no explosion ever took place, and none of the metals became ignited or inflamed.

The action even of potassium heated in the gas was not violent. But from the great absorption of gas, and from the precipitation of carbon indicated by the blackness produced, not only the new gas, but likewise the carbonic oxide, appeared to be decomposed.

The white oxide of zinc heated in the gas quickly decomposed it, just as readily indeed as the metal itself; there was

the same formation of the butter of zinc; but instead of carbonic oxide being produced, carbonic acid was formed; and as usual, there was no change of volume.

The protoxide of antimony fused in the gas rapidly decomposed it; the butter of antimony and the infusible peroxide were formed; there was no change of the volume of the gas, and the residual gas was carbonic oxide.

Sulphur and phosphorus sublimed in the gas, produced no apparent change; the volume of the gas was unaltered, and its characteristic smell was undiminished.

Mixed with hydrogene or oxygene singly, the gas was not inflamed by the electric spark, but mixed with both, in proper proportions, viz. two parts in volume of the former and one of the latter to two parts of the gas, a violent explosion was produced, and the muriatic and carbonic acid gasses were formed.

The gas transferred to water was quickly decomposed, the carbonic and muriatic acids were formed, as in the last experiment, and the effect was the same even when light was excluded.

From the mode of the formation of the gas and the condensation that takes place at the time, from the results of the decomposition of its ammoniacal salt, and from the analysis of the gas by metals and metallic oxides, it appears to be a compound of carbonic oxide and chlorine condensed into half the space which they occupied separately.

And from its combining with ammonia, and forming with this alkali a neutral salt, and from its reddening litmus, it seems to be an acid. It is similar to acids in other respects; in decomposing the dry sub-carbonate of ammonia, one part

in volume of it, expelling two parts of carbonic acid gas; and in being itself not expelled from ammonia by any of the acid gasses, or by acetic acid. Independent of these circumstances, were power of saturation to be taken as the measure of affinity, the attraction of this gas for ammonia must be allowed to be greater than that of any other substance, for its saturating power is greater; no acid condenses so large a proportion of ammonia; carbonic acid only condenses half as much, and yet does not form a neutral salt. The great saturating and neutralizing powers of this gas are singular circumstances, and particularly singular when compared with those of muriatic acid gas.

In consequence of its being decomposed by water, I have not been able to ascertain whether it is capable of combining with the fixed alkalies. Added to solutions of these substances it was absorbed, and carbonic acid gas was disengaged by an acid.

I have made the experiment on the native carbonates of lime and barytes, but the gas did not decompose these bodies. This indeed might be expected, since quick-lime, I find, does not absorb the gas: a cubic inch of it, exposed to the action of lime in a tube over mercury, was only diminished in two days to nine-tenths of a cubic inch, and no further absorption was afterwards observed to take place. But even this circumstance does not demonstrate that the gas has no affinity for lime, and is not capable of combining with it; for on making a similar experiment with carbonic acid, substituting this gas for the new compound, the result was the same; in two days only about one-tenth of a cubic inch was absorbed.

Though the gas is decomposed by water, yet it appears to

be absorbed unaltered by common spirits of wine, which contains so considerable a quantity of water; it imparted its peculiar odour to the spirit, and its property of affecting the eyes; five measures of the spirit condensed sixty measures of the gas.

It is also absorbed by the fuming liquor of arsenic, and by the oxymuriate of sulphur.

The former appeared to require for saturation ten times its own volume; six measures of the liquor condensed about sixty of the gas. The liquor thus impregnated was thrown into water, and a pretty appearance was produced by the sudden escape of bubbles of the gas; had not its intolerable smell convinced me that the gas was unaltered, I should not have conceived that it could pass through water undecomposed.

I cannot account for the assertion of M. M. GAY LUSSAC and THENARD and of Mr. MURRAY, that oxymuriatic gas does not, when under the influence of light, exert any action on carbonic oxide: I was inclined at first to suppose that the difference between their results and mine, might be owing to their not having exposed the gasses together to bright sunshine; but I have been obliged to relinquish this idea, since I have found that bright sun-shine is not essential, and that the combination is produced in less than twelve hours by the indirect solar rays, light alone being necessary.

The formation of the new gas may be very readily witnessed, by making a mixture of dry carbonic oxide and chlorine in a glass tube over mercury: if light be excluded, the chlorine will be absorbed by the mercury, the carbonic oxide alone remaining; but if bright sun-shine be immediately admitted when the mixture is first made, a rapid ascension of

the mercury will take place, and in less than a minute, the colour of the chlorine will be destroyed, and in about ten minutes the condensation will have ceased, and the combination of the two gasses will be complete.

It is requisite that the gasses should be dried for forming this compound; if this precaution is neglected, the new gas will be far from pure; it will contain a considerable admixture of the carbonic and muriatic acid gasses, which are produced in consequence of the decomposition of hygrometrical water. Indeed there is considerable difficulty in procuring the new gas tolerably pure; a good air pump is required, and perfectly tight stop-cocks, and dry gasses, and dry vessels.

I have endeavoured to procure the gas, by passing a mixture of carbonic oxide and chlorine through an earthen-ware tube heated to redness; but without success.

The specific gravity of the gas may be inferred from the specific gravities of its constituent parts jointly with the condensation that takes place at their union. According to Cruickshank, 100 cubic inches of carbonic oxide weigh 29,6 grains, and according to Sir Humphry Davy, 100 of chlorine are equal to 76,37 grains: hence as equal volumes of these gasses combine, and become so condensed as to occupy only half the space they before filled, it follows that 100 cubic inches of the new compound gas are equal to 105,97 grains. Thus this gas exceeds most others as much in its density as it does in its saturating power.

To ascertain whether chlorine has a stronger affinity for hydrogene than for carbonic oxide, I exposed a mixture of the three gasses in equal volumes to light. Both the new compound and muriatic acid gas were formed, and the affinities

were so nicely balanced, that the chlorine was nearly equally divided between them. And that the attraction of chlorine for both these gasses is nearly the same, appears to be confirmed by muriatic acid not being decomposed by carbonic oxide, or the new gas by hydrogene.

The chlorine and carbonic oxide are, it is evident from these last facts, united by strong attractions; and as the properties of the substance as a peculiar compound are well characterized, it will be necessary to designate it by some simple name. I venture to propose that of phosgene, or phosgene gas; from φως, light, and γινομαι, to produce, which signifies formed by light; and as yet no other mode of producing it has been discovered.

I have exposed mixtures consisting of different proportions of chlorine and carbonic acid to light, but have obtained no new compound.

The proportions in which bodies combine appear to be determined by fixed laws, which are exemplified in a variety of instances, and particularly in the present compound. Oxygene combines with twice its volume of hydrogene and twice its volume of carbonic oxide to form water and carbonic acid, and with half its volume of chlorine to form euchlorine; and chlorine reciprocally requires its own volume of hydrogene and its own volume of carbonic oxide to form muriatic acid and the new gas.

This relation of proportions is one of the most beautiful parts of chemical philosophy, and that which promises fairest, when prosecuted, of raising chemistry to the state and certainty of a mathematical science.

VII. *A Narrative of the Eruption of a Volcano in the Sea off the Island of St. Michael. By* S. Tillard, *Esq. Captain in the Royal Navy. Communicated by the Right Hon. Sir* Joseph Banks, *Bart. K. B. P. R. S.*

Read February 6, 1812.

Approaching the island of St. Michael's, on Sunday the 12th of June, 1811, in His Majesty's Sloop Sabrina, under my command, we occasionally observed, rising in the horizon, two or three columns of smoke, such as would have been occasioned by an action between two ships, to which cause we universally attributed its origin. This opinion was, however, in a very short time changed, from the smoke increasing and ascending in much larger bodies than could possibly have been produced by such an event, and having heard an account, prior to our sailing from Lisbon, that in the preceding January or February a volcano had burst out within the sea near St. Michael's, we immediately concluded that the smoke we saw proceeded from that cause, and on our anchoring the next morning in the road of Ponta del Gada, we found this conjecture correct as to the cause, but not to the time; the eruption of January having totally subsided, and the present one having only burst forth two days prior to our approach, and about three miles distant from the one before alluded to.

Desirous of examining as minutely as possible a contention so extraordinary between two such powerful elements, I set

off from the city of Ponta del Gada on the morning of the 14th, in company with Mr. Read, the Consul General of the Azores, and two other gentlemen. After riding about twenty miles across the NW. end of the island of St. Michael's, we came to the edge of a cliff from whence the volcano burst suddenly upon our view in the most terrific and awful grandeur. It was only a short mile from the base of the cliff, which was nearly perpendicular, and formed the margin of the sea; this cliff being as nearly as I could judge from three to four hundred feet high. To give you an adequate idea of the scene by description is far beyond my powers; but for your satisfaction I shall attempt it.

Imagine an immense body of smoke rising from the sea, the surface of which was marked by the silvery ripling of the waves, occasioned by the light and steady breezes incidental to those climates in summer. In a quiescent state, it had the appearance of a circular cloud revolving on the water like an horizontal wheel, in various and irregular involutions, expanding itself gradually on the lee side, when suddenly a column of the blackest cinders, ashes, and stones would shoot up in form of a spire at an angle of from ten to twenty degrees from a perpendicular line, the angle of inclination being universally to windward: this was rapidly succeeded by a second, third, and fourth, each acquiring greater velocity, and overtopping the other till they had attained an altitude as much above the level of our eye, as the sea was below it.

As the impetus with which the columns were severally propelled diminished, and their ascending motion had nearly ceased, they broke into various branches resembling a groupe of pines, these again forming themselves into festoons of white

feathery smoke in the most fanciful manner imaginable, intermixed with the finest particles of falling ashes, which at one time assumed the appearance of innumerable plumes of black and white ostrich feathers surmounting each other; at another, that of the light wavy branches of a weeping willow.

During these bursts, the most vivid flashes of lightning continually issued from the densest part of the volcano; and the cloud of smoke now ascending to an altitude much above the highest point to which the ashes were projected, rolled off in large masses of fleecy clouds, gradually expanding themselves before the wind in a direction nearly horizontal, and drawing up to them a quantity of water spouts, which formed a most beautiful and striking addition to the general appearance of the scene.

That part of the sea, where the volcano was situated, was upwards of thirty fathoms deep, and at the time of our viewing it the volcano was only four days old. Soon after our arrival on the cliff, a peasant observed he could discern a peak above the water: we looked, but could not see it; however, in less than half an hour it was plainly visible, and before we quitted the place, which was about three hours from the time of our arrival, a complete crater was formed above the water, not less than twenty feet high on the side where the greatest quantity of ashes fell; the diameter of the crater being apparently about four or five hundred feet.

The great eruptions were generally attended with a noise like the continued firing of cannon and musquetry intermixed, as also with slight shocks of earthquakes, several of which having been felt by my companions, but none by myself, I had become half sceptical, and thought their opinion arose

merely from the force of imagination; but while we were sitting within five or six yards of the edge of the cliff, partaking of a slight repast which had been brought with us, and were all busily engaged, one of the most magnificent bursts took place which we had yet witnessed, accompanied by a very severe shock of an earthquake. The instantaneous and involuntary movement of each was to spring upon his feet, and I said "this admits of no doubt." The words had scarce passed my lips, before we observed a large portion of the face of the cliff, about fifty yards on our left, falling, which it did with a violent crash. So soon as our first consternation had a little subsided, we removed about ten or a dozen yards further from the edge of the cliff, and finished our dinner.

On the succeeding day, June 15th, having the Consul and some other friends on board, I weighed, and proceeded with the ship towards the volcano, with the intention of witnessing a night view; but in this expectation we were greatly disappointed, from the wind freshening and the weather becoming thick and hazy, and also from the volcano itself being clearly more quiescent than it was the preceding day. It seldom emitted any lightning, but occasionally as much flame as may be seen to issue from the top of a glass-house or foundery chimney.

On passing directly under the great cloud of smoke, about three or four miles distant from the volcano, the decks of the ship were covered with fine black ashes, which fell intermixt with small rain. We returned the next morning, and late on the evening of the same day, I took my leave of St. Michael's to complete my cruize.

On opening the volcano clear of the NW. part of the island,

after dark on the 16th, we witnessed one or two eruptions that, had the ship been near enough, would have been awfully grand. It appeared one continued blaze of lightning; but the distance which it was at from the ship, upwards of twenty miles, prevented our seeing it with effect.

Returning again towards St. Michael's on the 4th of July, I was obliged, by the state of the wind, to pass with the ship very close to the island, which was now completely formed by the volcano, being nearly the height of Matlock High Tor, about eighty yards above the sea. At this time it was perfectly tranquil, which circumstance determined me to land, and explore it more narrowly.

I left the ship in one of the boats, accompanied by some of the officers. As we approached, we perceived that it was still smoking in many parts, and upon our reaching the island found the surf on the beach very high. Rowing round to the lee side, with some little difficulty, by the aid of an oar, as a pole, I jumped on shore, and was followed by the other officers. We found a narrow beach of black ashes, from which the side of the island rose in general too steep to admit of our ascending; and where we could have clambered up, the mass of matter was much too hot to allow our proceeding more than a few yards in the ascent.

The declivity below the surface of the sea was equally steep, having seven fathoms water, scarce the boat's length from the shore, and at the distance of twenty or thirty yards, we sounded twenty-five fathoms.

From walking round it, in about twelve minutes, I should judge that it was something less than a mile in circumference; but the most extraordinary part was the crater, the mouth of

which, on the side facing St. Michael's, was nearly level with the sea. It was filled with water, at that time boiling, and was emptying itself into the sea, by a small stream about six yards over, and by which I should suppose it was continually filled again at high water. This stream, close to the edge of the sea, was so hot, as only to admit the finger to be dipped suddenly in, and taken out again immediately.

It appeared evident, by the formation of this part of the island, that the sea had, during the eruptions, broke into the crater in two places, as the east side of the small stream was bounded by a precipice, a cliff between twenty and thirty feet high forming a peninsula of about the same dimensions in width, and from fifty to sixty feet long, connected with the other part of the island by a narrow ridge of cinders and lava, as an isthmus of from forty to fifty feet in length, from which the crater rose in the form of an amphitheatre.

This cliff, at two or three miles distance from the island, had the appearance of a work of art resembling a small fort or block house. The top of this we were determined, if possible, to attain; but the difficulty we had to encounter in doing so was considerable; the only way to attempt it was up the side of the isthmus, which was so steep, that the only mode by which we could effect it, was by fixing the end of an oar at the base, with the assistance of which we forced ourselves up in nearly a backward direction.

Having reached the summit of the isthmus, we found another difficulty, for it was impossible to walk upon it, as the descent on the other side was immediate, and as steep as the one we had ascended; but by throwing our legs across it, as would be done on the ridge of a house, and moving ourselves

forward by our hands, we at length reached that part of it where it gradually widened itself and formed the summit of the cliff, which we found to have a perfectly flat surface, of the dimensions before stated.

Judging this to be the most conspicuous situation, we here planted the Union, and left a bottle sealed up containing a small account of the origin of the island, and of our having landed upon it, and naming it Sabrina Island.

Within the crater I found the complete skeleton of a guard-fish, the bones of which being perfectly burnt, fell to pieces upon attempting to take them up; and by the account of the inhabitants on the coast of St. Michael's, great numbers of fish had been destroyed during the early part of the eruption, as large quantities, probably suffocated or poisoned, were occasionally found drifted into the small inlets or bays.

The island, like other volcanic productions, is composed principally of porous substances, and generally burnt to complete cinders, with occasional masses of a stone, which I should suppose to be a mixture of iron and lime-stone; but have sent you specimens to enable you to form a better judgment than you possibly can by any description of mine.

VIII. *On the primitive Crystals of Carbonate of Lime, Bitter-Spar, and Iron-Spar. By* William Hyde Wollaston, *M.D. Sec. R.S.*

Read February 13, 1812.

When I formerly described to the Society a goniometer* upon a new construction for measuring the angles of crystals, I expressed an expectation that we should thereby be enabled to correct former observations made by means of less accurate instruments. I took occasion to mention one instance of inaccurate measurement in the primitive angle of the common carbonate of lime; and I have had the satisfaction to find the necessity of a correction, in that instance, confirmed by Mons. Malus, and admitted by the Abbé Haüy, in a work † published nearly at the same time.

It is by no means my design to detract in any degree from the merit of that justly celebrated crystallographer, to the surprising accuracy of whose measurements I could, in various instances, bear testimony. I hope, on the contrary, that in bringing forward two more observations similar to the preceding, and intimately connected with it, I shall offer what will not only appear interesting to crystallographers in general, but will be peculiarly gratifying to the Abbé Haüy.

In his Traité de Minéralogie, and again more recently in his Tableau Comparatif, the same primitive form is assigned

* Phil. Trans. 1809.

† Tableau Comparatif des resultats de la Crystallographie et de l'Analyse Chimique,

to three substances very different in their composition, to carbonate of lime, to magnesian carbonate of lime (or bitter-spar) and to carbonate of iron.

It has been objected to Mons. HAÜY, that according to his method identity of form should be accompanied by identity of composition, unless the form were one of the common regular solids. For though in that case any geometrician would readily admit it to be very probable, that many different substances might concur in assuming the same form of cube, of octohedron, or of dodecahedron, &c. there does not appear a corresponding probability that any two dissimilar substances would assume the same form of a particular rhomboid of 105° and a few minutes, to which no such geometric regularity or peculiar simplicity can be ascribed.

But though so accurate a correspondence, as has been hitherto supposed to exist in the measures of the three carbonates above-mentioned, might be justly considered as highly improbable, no degree of improbability whatever attaches to the supposition, that their angles approach each other by some difference, so small as hitherto to have escaped detection. And this in fact I find to be the case.

Since the angles observable in *fractures* of crystalline substances are subject to vary a little at different surfaces, and even in different parts of the same surface (as is evident from the confused image seen by reflection from them), I shall not at present undertake to determine the angles of these bodies to less than five minutes of a degree. This, indeed, is the smallest division of the goniometer that I usually employ, as I purposely decline giving so much time to these inquiries, as would be requisite for attempting to arrive at greater precision.

The most accurate determination of the angle of carbonate of lime is probably that of Mons. MALUS, who measured it by means of a repeating circle, and found it to be 105° 5′. And this, indeed, is the result to which I formerly came by a different method.* If it differ in any respect from this quantity, I am inclined to think that it will more likely be found to be deficient by a few minutes, than to exceed the measure here assigned; and accordingly to differ still more widely from those angles which I am about to mention.

In the magnesian carbonate of lime, or bitter-spar, the primitive form is well known to be a regular rhomboid, as well as that of carbonate of lime, and so nearly resembling it, as to have been hitherto supposed the same. I find, however, a difference of 1° 10′ in the measures of these crystals; for that of the magnesian carbonate is full 106¼°, as I have observed with uniformity in at least five different specimens of this substance obtained from situations very distant from each other.

The primitive angle of iron-spar is still more remote from that of the carbonate of lime, which it exceeds by nearly two degrees. I have examined various specimens of this substance, some pure white, others brown, some transparent, others opake. That which gives the most distinct image by reflection is of a brownish hue, with the semi-transparency of horn. It was obtained from a tin mine, called Maudlin Mine, near Lostwithiel in Cornwall. By repeated measurement of small fragments of this specimen, the angle appears to be so nearly 107°, that I cannot form any judgment whether in perfect crystals it will prove to be greater or less than that angle.

In this instance the carbonate of iron is nearly pure, and so

* Phil. Trans. 1802, p. 385.

perfectly free from carbonate of lime, as to render it highly probable that in other specimens, having the same angle, but containing also carbonate of lime or other ingredients intermixed, the form is really dependent on the carbonate of iron alone.

It appears, however, not unlikely that when substances, which agree so nearly in their primitive angle, are intermixed in certain proportions, they may each exert their power; and may occasion that confused appearance of crystallization with curved surfaces, known by the name of pearl-spar. I cannot say that I have made any accúrate comparative analyses which may be adduced in support of the hypothesis, that mixtures are more subject to cúrvature than pure chemical compounds; but it is very evident, from the numerous analyses that have been made of iron-spar by other chemists, how extremely variable they are in their composition, and consequently how probable it is, that the greater part of them are to be regarded as mixtures; although it be also possible, that there may exist a triple carbonate of lime and iron as a strict chemical compound.

It seems not unlikely, that there may hereafter be found some carbonate allied to the preceding, which may owe its form to the presence of manganese; but notwithstanding the liberality which happily prevails in general among those who have it in their power to assist in such inquiries, I have not had the good fortune to meet with any such compound; and I am unwilling, merely in the hope of making such an addition, any longer to defer communicating an observation, which I hope will be of real utility in the discrimination of bodies that differ so essentially in their composition.

IX. *Observations intended to show that the progressive Motion of Snakes is partly performed by means of the Ribs. By* Everard Home, *Esq. F. R. S.*

Read February 27, 1812.

On a former occasion I laid before the Society a description of the mechanism of the hood of the cobra de capello snake of the East Indies, the coluber naja of Linnæus, in which the ribs of the neck are shewn to be formed in a particular manner; so that when they are raised, the skin becomes stretched out, and puts on the appearance of a hood.

The ribs so employed have several peculiarities, which, I took for granted, were confined to those of the neck, for I was not in possession of the bodies of the snakes, and therefore could not examine the others; but have since found that many of these peculiarities are not only common to all the ribs of this snake, but to those of the whole tribe.

This fact, as it escaped my observation at that time, would have still done so, had it not been for the following circumstances.

A coluber of unusual size, lately brought to London to be exhibited, was shewn to Sir Joseph Banks; the animal was lively, and moved along the carpet briskly: while it was doing so, Sir Joseph thought he saw the ribs come forward in succession like the feet of a caterpillar. This remark he immediately communicated to me, and gave me an opportunity of seeing the snake and making my own observations.

The fact was readily established, and I could feel the ribs with my fingers as they were brought forward; when a hand was laid flat under the snake, the ends of the ribs were distinctly felt upon the palm, as the animal passed over it.

This becomes a more interesting discovery, as it constitutes a new species of progressive motion, and one widely different from those already known.

In the draco volans the ribs form the skeleton of the wings, by means of which the animal flies, the five posterior ribs being bent backwards and elongated for that purpose, so that in that instance the progressive motion is performed by the ribs, but those particular ribs are superadded for this purpose, and make no part of the organs of respiration; whereas in the snake the ribs are so constructed, as to perform their office with respect to the lungs as well as progressive motion.

That ribs are not essential to the breathing of all animals, whose lungs are situated in the same manner as in snakes, is proved by the syren having no ribs; but as this animal has also gills, and can breathe in water as well as in air, the lungs are not so constantly employed, and probably a less perfect supply of air to them may suffice.

In animals in general, the ribs are articulated to the back bone by means of a convex surface, which moves upon a slightly concave one formed upon two of the vertebræ, partly on the one and partly on the other, so that there is a rib situated between every two vertebræ of the back; but in the snake tribe, the head of the rib has two slightly concave surfaces which move upon a convex protuberance belonging to each vertebra, so th a there is a rib to each of the vertebræ.

One advantage of this peculiarity is, that it prevents the ribs from interfering with the motion of the vertebræ on one

another. The vertebræ are articulated together by ball and socket joints (the ball being formed upon the lower end and the socket on the upper one), and have therefore much more extensive motion than in other animals.

The muscles, which bring the ribs forward, consist of five sets, one from the transverse process of each vertebra to the rib immediately behind it, which rib is attached to the next vertebra. The next set goes from the rib a little way from the spine just beyond where the former terminates, it passes over two ribs, sending a slip to each, and is inserted into the third: there is a slip also connecting it with the next muscle in succession. Under this is the third set, which arises from the posterior side of each rib, passes over two ribs, sending a lateral slip to the next muscle, and is inserted into the third rib behind it.

The fourth set passes from one rib over the next, and is inserted into the second rib.

The fifth set goes from rib to rib.

On the inside of the chest there is a strong set of muscles attached to the anterior surface of the vertebræ, and passing obliquely forwards over four ribs to be inserted into the fifth rib, nearly at the middle part between the two extremities.

From this part of each rib a strong flat muscle comes forward on each side over the viscera, forming the abdominal muscles, and uniting in a beautiful middle tendon, so that the lower half of each rib, which is beyond the origin of this muscle, and which is only laterally connected to it by loose cellular membrane, is external to the belly of the animal for the purpose of progressive motion; and that half of each rib next the spine, as far as the lungs extend, is employed in respiration.

At the termination of each rib is a small cartilage in shape corresponding to the rib, only tapering to the point. Those of the opposite ribs have no connection, and when the ribs are drawn outwards by the muscles, are separated to some distance, and rest through their whole length on the inner surface of the abdominal scuta, to which they are connected by a set of short muscles: they have also a connection with those of the neighbouring ribs by a set of short straight muscles.

These observations apply to snakes in general; but they have been particularly examined in a boa constrictor, three feet nine inches long, preserved in the Hunterian Museum. In all snakes, the ribs are continued to the anus, while the lungs, seldom occupy more than one-half of the extent of the cavity covered by the ribs. These lower ribs can only be employed for the purpose of progressive motion, and therefore correspond in that respect with the ribs in the draco volans superadded to form the wings.

The parts of which a description has been attempted, will be better understood by an inspection of Plates IV. and V. than by any explanation that words can convey.

In Plate VI. the joints between the vertebræ and ribs are represented of the natural size from the skeleton of a large boa, sent from the East Indies by the late Sir WILLIAM JONES and deposited in the Hunterian Museum. On the under surface of the vertebra is a protuberance for the attachment of muscles peculiar to this genus, it varies in size in the different species, and explains the power attributed to the boa constrictor.

When the snake is going to put itself in motion, the ribs of the opposite sides are drawn apart from each other, and the small cartilages at the ends of them are bent upon the upper surfaces of the abdominal scuta, upon which the ends of the

ribs rest; and as the ribs move in pairs, the scutum under each pair is carried along with it. This scutum, by its posterior edge, lays hold of the ground and becomes a fixed point from whence to set out anew. This motion is beautifully seen when a snake is climbing over an angle to get upon a flat surface.

When the animal is moving, it alters its shape from a circular or oval form, to something approaching to a triangle, of which the surface on the ground forms the base.

The coluber and boa having large abdominal scuta, which may be considered as hoofs or shoes, are the best fitted for this kind of progressive motion; there is, however, a similar structure of ribs and muscles in the anguis and amphisbæna.

In the anguis the ribs are proportionally weaker, and as these have nothing to correspond with the scuta, it is probable this mode of progressive motion is less necessary to them.

The rings of the amphisbæna are a near approach to the large scuta.

DESCRIPTION OF THE PLATES.

Plate IV.

A lateral view of the muscles of the boa constrictor.

A A. The straight muscles of the back.

B B. The first set of muscles which arises from the transverse process of each vertebra, and is inserted into the rib behind it close to its head.

C C. The second set.

D D. The third set.

E E. The fourth set.

F F. The fifth set.

G G. Short muscles which pass from cartilage to cartilage.

H H. A set of oblique muscles which passes from the anterior side of the bony extremity of each rib to the posterior edge of each scutum.

I I. Muscles which pass from the ribs near their heads obliquely backwards, to be inserted into the skin at the edge of each scutum.

K. Muscles of the scuta.

PLATE V.

An internal view of the abdominal muscles of the boa constrictor.

A A. The muscles which pass from cartilage to cartilage of the different ribs.

B B. A set of muscles which passes from the point of each rib over two ribs to the middle of the third.

C C. A similar set of muscles continued from the opposite side of the rib, passing over three ribs to the body of the vertebra.

D D. The abdominal muscles which arise from the anterior edge of each rib, and pass to the linea alba.

E E. The linea alba.

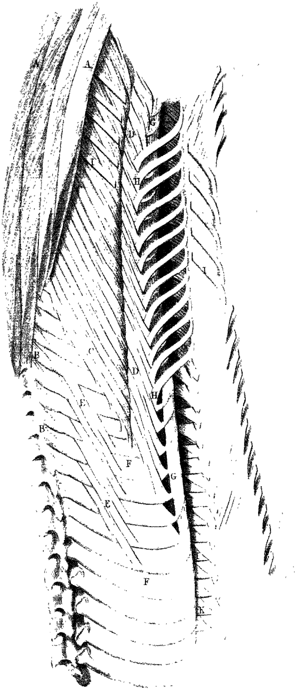
F F. The termination of the set of oblique muscles which passes from the bony extremities of the ribs to the edges of the scuta.

G G. The muscles of the scuta consisting of two sets which decussate each other.

PLATE VI.

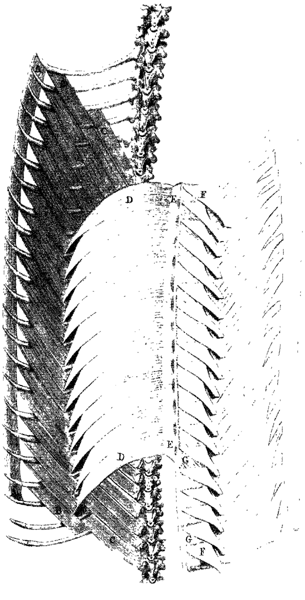
Represents two vertebræ and portions of ribs of the large boa to show their articulating surfaces.

a a. The process peculiar to the vertebra of the boa.

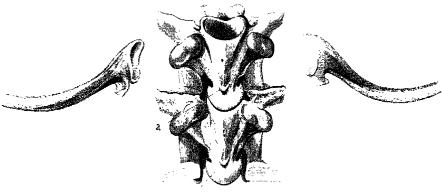


W. Clift del.

J. Basire sc.



W. Clift del. J. Basire sc.



H. Clift del.   *J. Basire sc.*

X. *An Account of some Experiments on the Combinations of different Metals and Chlorine, &c. By* John Davy, *Esq. Communicated by Sir* Humphry Davy, *Knt. LL.D. Sec. R.S.*

Read February 27, 1812.

Introduction.

My brother, Sir Humphry Davy, appears to me to have demonstrated, in his last Bakerian Lecture, the existence of a class of bodies similar to metallic oxides, and consisting of metals in union with chlorine or oxymuriatic acid.

These combinations are the principal subject of the following pages. I shall do myself the honour of giving an account of the experiments I have made to ascertain the proportions of their constituent parts, and likewise of describing some that have not yet been noticed.

I shall have to relate also the attempts I have made to ascertain the proportions of sulphur in several sulphurets, and the experiments I have performed to estimate the quantity of oxygene in some metallic oxides. The general analogy of definite proportions led me to both these undertakings. This analogy, it will be perceived, I have constantly kept in view, and have had recourse to, both for detecting inaccuracies in my own experiments, and in considering the results of the experiments of others.

As the nomenclature connected with the old hypothesis, respecting oxymuriatic acid, is inconsistent with the new views

of this substance, I shall venture to call the compounds of the metals and chlorine to be treated of, by the names which my brother has proposed for them.

1. *On the Combinations of Chlorine and Copper, &c.*

There are two distinct combinations of chlorine and copper, both of which may be directly made by the combustion of this metal in chlorine gas. When the gas was admitted into an exhausted retort containing copper filings, the filings became ignited, a fixed fusible substance quickly formed, and the interior of the retort soon became lined with a fine yellowish brown sublimate. The former substance evidently contains least chlorine, for when it was heated alone in chlorine gas, it absorbed an additional portion, and was converted into the latter. Hence the fixed compound may, in conformity with the principles of Sir Humphry Davy's nomenclature, be called cuprane, and the yellow sublimate, cupranea.

Cuprane may be procured in several other ways. It may be obtained by heating together copper filings and corrosive sublimate; and it was thus first discovered by Boyle, who called it resin of copper, from its similitude to common resin. Two parts of corrosive sublimate, and one part of copper filings, I have found the best proportions of the materials.

It may be obtained by boiling copper filings in muriatic acid, or by exposing slips of copper partially immersed in this acid to the atmosphere. In the last instance, I have found the changes connected with the formation of cuprane rather complicated; the copper exposed receives oxygene from the atmosphere, and acid from the ascending muriatic acid fumes, and is thus converted into a green insoluble salt, and this

absorbing more muriatic acid, slowly passes into the deliquescent muriat, which flowing into the muriatic acid is changed by the action of the immersed copper into cuprane.

M. Proust, the first modern chemist who examined cuprane, and who is commonly considered as the first discoverer of this compound, found it produced by the action of muriat of tin on muriat of copper; he named it white muriat of copper, and ascertained that a similar substance results from the decomposition of the common deliquescent muriat by heat.

Cuprane, by whatever means prepared, possesses the same properties. It is fusible at a heat just below that of redness, and in a close vessel, or a vessel with a very small orifice, it is not decomposed or sublimed by a strong red heat; but if air, on the contrary, is freely admitted, it is dissipated in dense white fumes. It is insoluble in water. It effervesces in nitric acid. It silently dissolves in muriatic acid, from which it may be separated by the addition of water, which precipitates it unaltered; and it is decomposed by a solution of potash; or by heating it with the fused hydrated alkali: when it affords the orange oxide of copper. Its colour, transparency, and texture appear alone to vary. It is generally opaque, of a dark brown colour, and of a confused hackly texture; but I have obtained it by cooling it slowly after it has been strongly heated, of a light yellow colour, semi-transparent, and crystallized, apparently in small plates.

Cupranea is only very slowly formed by heating cuprane in chlorine gas. The best mode that I have found, of procuring it, is by slowly evaporating to dryness, at a temperature not much above 400 of Fahrenheit, the deliquescent muriat of copper. Thus made, it has the same appearance

and the same properties, as when directly formed. It is of a yellow colour, and pulverulent. Exposed to the atmosphere, it is converted, by the action and absorption of water, into the deliquescent muriat, and its colour, during this alteration, changes from yellow first to white, and lastly to green. It is decomposed by heat; and even in chlorine gas when the experiment is made on a pretty large quantity, part of the chlorine is expelled, and assumes the gaseous state, and cuprane remains.

I have employed the same methods for ascertaining the proportions of the constituent parts of both these combinations. I have separated the copper by iron, and the chlorine by means of nitrat of silver.

A solution of 80 grains of cuprane in nitro-muriatic acid, precipitated by iron, afforded 51.2 grains of copper, well washed, and perfectly dried.

A solution of the same quantity of cuprane in nitric acid, precipitated by nitrat of silver, afforded 117.5 grains of horn silver dried, till it ceased to suffer any loss of weight by exposure to a temperature above 500 Fahrenheit.

Since horn silver contains 24.5 per cent. of chlorine,* 80 grains of cuprane appear to contain 51.2 grains of copper and 28.8 of chlorine. And 100 appear to consist of

36 chlorine
64 copper

100

* This I have ascertained by synthesis; 12 grains of pure silver dissolved in nitric acid, and precipitated with muriat of ammonia, yielded 15.9 grains of fused horn silver. I do not give the particulars of the experiment, which was very carefully made; because the result very nearly agrees with that of Klaproth, and of other chemists.

A solution of 40 grains of cupranea in water, acidulated with muriatic acid, precipitated by iron afforded 18.8 grains of copper.

And a solution of 20 grains of cupranea in water, precipitated by nitrat of silver, afforded 43 grains of horn silver.

Hence 100 of cupranea, omitting the very slight loss, appear to consist of

53	chlorine
47	copper
100	

The deliquescent muriat and the native muriat of copper of Peru, belong to a class of compounds apparently distinct from the preceding combinations of copper and chlorine.

The deliquescent salt is well understood; and its composition may be inferred, independent of its water, from that of cupranea.

The native muriat is less known, I shall therefore relate the experiments I have made on this interesting mineral.

The specimen I have examined is part of a very fine one, presented to Sir HUMPHRY DAVY by WILLIAM JACOB, Esq. M. P. and deposited in the Museum of the Royal Institution. It consists of muriat and carbonat of copper, of red oxide of iron, and of green coloured quartz. The muriat is partly crystallized; the crystals, from the trials I have made of them, appeared to be pure, and they were, on that account, made the subject of my experiments.

The crystallized muriat dissolves entirely and without effervescence, in all the acids in which I have tried it, and the deliquescent muriat of copper is in each instance formed, and

a combination of brown oxide of copper with the acid employed.

Heated slowly in a bent luted glass tube, connected with mercury, the native muriat affords water and oxygene gas, and the residue is an agglutinated brownish mass, which dissolves in muriatic acid and gives a greenish precipitate with potash, and is apparently a mixture of brown oxide of copper and cuprane. When the heat is raised rapidly to redness, the water expelled is impregnated with muriatic acid, and muriat of copper. I have obtained from 25 grains of the mineral heated to redness till gas ceased to be produced, just two cubic inches of oxygene. This expulsion of oxygene seems to be owing to the action of chlorine on the brown oxide to form cuprane; and there is, I have ascertained, a similar production of oxygene when heat is applied to a mixture of the deliquescent muriat and brown oxide of copper.

From these results, which perfectly agree with those obtained by eminent chemists on the Continent, who have examined different specimens of this mineral, it appears to be a submuriat of copper, differing in a chemical point of view from the deliquescent salt, merely in containing a smaller proportion of acid.

The following experiments were made with the design of ascertaining the proportions of its constituent parts.

50 grains of the crystals in powder, boiled in a solution of 50 grains of potash, afforded 36.5 grains of brown oxide of copper heated to dull redness.

And 20 grains dissolved in nitric acid, and precipitated by means of nitrat of silver afforded 12.9 grains of dry horn silver.

Hence, considering the deficiency of weight, as indicating the quantity of combined water, 100 of the native sub-muriat of copper seem to consist of

$$\begin{array}{l} 73.0 \text{ brown oxide} \\ 16.2 \text{ muriatic acid} = \\ 10.8 \text{ water} \end{array} \left\{ \begin{array}{l} 15.8025 \text{ chlorine} \\ \\ \quad .47 \quad \text{hydrogene.} \end{array} \right.$$

This analysis, allowance being made for difference of theory, nearly agrees with that of KLAPROTH.

M. PROUST, I believe, first discovered an artificial compound similar to the native sub-muriat of copper. He obtained it, in the preparation of the nitro-muriat of copper, and also by a partial abstraction of the acid of the deliquescent muriat, by means of an alkali. I have found that it may be procured in several other ways. It may be made directly by adding the hydrated blue oxide of copper to a solution of muriat of copper; and it may be very readily and economically prepared, by exposing to the atmosphere slips of copper partially immersed in muriatic acid; and it is also produced by the exposure of cuprane to the atmosphere. Its production in the last instance is accompanied with that of the deliquescent muriat; and the formation of both seems to be owing to the absorption of water and oxygene; for cuprane, I have found, though apparently not in the least acted on by dry oxygene gas, is quickly changed when moistened with water and confined in a jar of this gas, and there is a rapid absorption of oxygene.*

I have not examined all the specimens obtained by these different methods minutely, though sufficiently, I conceive, to

* I have been informed that submuriat of copper is sometimes found in the neighbourhood of volcanoes, particularly in that of Vesuvius. By means of the above facts, it is evident that its production might be accounted for in such situations.

ascertain their identity, and their similarity to the native compound. The colour of all of them is greenish white, like that of the native, in a finely divided state. When heated, they all afford water, oxygene gas, and a mixture of cuprane and brown oxide of copper.

I have analysed only the submuriat, precipitated from a solution of muriat of copper, by a weak solution of potash.

50 grains of this, well washed and dried, boiled in a solution of potash, afforded 36.3 grains of dried brown oxide of copper.

And 20 grains dissolved in nitric acid, and precipitated by nitrat of silver, afforded 12.75 grains of dried horn silver. These results differ so little from those obtained with the native, as fairly to permit the conclusion, that the composition of the artificial and native submuriat of copper is the same.

2. *On the Combinations of Tin and Chlorine, &c.*

Tin, like copper, is capable of combining with two different proportions of chlorine. The liquor of Libavius, one of the combinations, is directly formed by the combustion of the metal in chlorine gas; and the other, I find, may be produced by heating together an amalgam of tin and calomel. Thus obtained, it is similar to that which may be procured, by evaporating to dryness, the muriat containing the gray oxide of tin, and fusing the residue in a close vessel. Both are of a gray colour, and of a resinous lustre and fracture, and both inflame, like tin itself, when heated in chlorine gas, and are converted into the liquor of Libavius by the absorption of a fresh portion of chlorine. Hence, as the liquor of Libavius

contains the largest proportion of chlorine, it may be called stannanea, and the other compound stannane.

Stannane is fusible at a heat below that of dull redness; it bears this temperature, if air be nearly excluded, without undergoing any change; but when subjected to a heat, as strong as glass will bear without being fused, it appears to be, from the slight fume produced, partially decomposed.

It affords the liquor of Libavius when heated with corrosive sublimate, nitre, red oxide of mercury, or with the hyperoxymuriat of potash. In the three last instances, oxide of tin is also formed; and with the hyperoxymuriat, the action is so violent, that inflammation is actually produced.

The liquor of Libavius and aurum musivum are formed when stannane is heated with sulphur.

Stannane, by the action of water, appears to be converted into the insoluble submuriat of tin, and the acidulous muriat.

The stannanea or liquor of Libavius, that I have examined, was made by heating together an amalgam of tin and corrosive sublimate, in the proportions commonly recommended. I have obtained this compound in another way, by treating the concentrated solution of the peroxide of tin in muriatic acid, with strong sulphuric acid; a gentle heat applied to this mixture contained, in a retort, expels the fuming liquor, which may be condensed, as usual, in a cold receiver.

The only new and remarkable property, which I have observed the liquor of Libavius to possess, is, its action on oil of turpentine. I was led to make trial of it from an idea of Sir Humphry Davy, that the combinations of the metals and chlorine might be soluble in oils. In the first experiment, when I poured the fuming liquor into the oil, inflammation immediately took

place, with violent ebullition and production of dense reddish fumes. I have used other specimens of oil of turpentine, expecting a similar inflammation, but without its occurrence, though there has been in every instance a considerable action. The mixture of the two being made in a retort connected with mercury, no gas was generated, oxide of tin appeared to be formed, and a viscid oil was produced, which, like the fat oils, left a permanent stain on paper, and had little smell or taste, and which, digested with alcohol, imparted something which occasioned a permanent cloudiness on the admixture of water, and an odour to me not unlike that of artificial camphor. The action of the liquor of Libavius on the oil of turpentine is worthy of further inquiry. The preceding account of it, I am aware is very incomplete; but I trust it will serve to call the attention of chemists to a subject so curious.

To discover the proportions of tin, and consequently of chlorine in stannane and stannanea, I have taken advantage of the superior affinity of zinc for chlorine, by means of which the tin is separated in its metallic state.

69.5 grains of stannane, made by heating in a glass tube with a very small orifice, an amalgam of tin with calomel, were, with the exception of two grains of metallic mercury, apparently a mere mechanical mixture, entirely dissolved in dilute muriatic acid. A slip of clean zinc, immersed in this solution decanted from the residual mercury, quickly precipitated the tin in a very beautiful plumose form; and this precipitate collected on a filter, and well washed and dried and fused into one globule under a cover of tallow in a small glass tube, weighed 42 grains.

As therefore 67.5 grains of stannane contain 42 grains of tin, 100 appear to consist of

62.22	tin
37.78	chlorine
100.00	

As stannanea is extremely volatile, it is difficult to weigh it, with perfect accuracy. The mode I adopted, was to pour it into a bottle half full of water, the weight of which was previously ascertained, and to infer the quantity added by the increase of weight.

81.75 grains of stannanea thus weighed in water,* afforded when decomposed by zinc 34 grains of tin.

Hence 100 of stannanea appear to be composed of

42.1	tin
57.9	chlorine
100.0	

I am not acquainted with any analytical method for directly ascertaining the proportion of chlorine in either of the two preceding combinations. Nitrat of silver, when immediately applied, will not answer the purpose, because the oxide of silver is partially reduced by the solution of stannane; and an oxide of tin is thrown down in mixture with the horn silver from the liquor of Libavius.

* A little muriatic acid was added before the zinc was introduced, to dissolve the oxide of zinc, which, in other similar experiments, I observed was rapidly formed, and which, from the large quantity of hydrogene evolved, appeared to be owing to the decomposition of water, chiefly in consequence of the Galvanic effect of the contact of the two different metals, zinc and tin.

M. Proust, to whom we are indebted for very excellent investigations of the different combinations of copper and tin, first discovered a submuriat of tin. He found that a solution of potash precipitated from the solution of muriat of tin this compound, and not the pure gray oxide of tin.

I have obtained it by his method, and all its properties which I have observed, are perfectly agreeable to its supposed composition.

It is decomposed by a red heat. Subjected to distillation in a small bent glass tube connected with mercury, no gas was produced, water containing muriatic acid and muriat of tin was expelled, and a sublimate like stannane was formed, and the fixed residue was gray oxide of tin.

It effervesces violently with nitric acid; and strong sulphuric acid expels from it muriatic acid fumes.

It dissolves without effervescence in the muriatic and acetic and in the dilute, nitric, and sulphuric acids; and all these acid solutions, as they give a black precipitate with a solution of corrosive sublimate, appear to contain the tin in the state of gray oxide.

The complete analysis of this submuriat of tin is difficult. The oxide it contains cannot be accurately separated by potash, nor can nitrat of silver be employed to ascertain the proportion of muriatic acid.

I have found 50 grains of it, dissolved in muriatic acid, to afford, when precipitated by zinc, 31 grains of metallic tin. Now as this submuriat is similar to the submuriat of copper, the analogy being imperfect only in the latter containing the peroxide, and the former the protoxide, it is natural to infer that the proportion of muriatic acid is similar in both. But the

proportion of muriatic acid in the submuriat of copper is apparently half of that which exists in the muriat; hence, supposing the composition of the submuriat of tin to be similar, 100 of it will consist of

70.4	gray oxide
19.0	muriatic acid
10.6	water
100.0	

Probability alone can be attached to this estimate. I have not given the calculations by which it was made, as their data are liable to objection.

3. *On the Combinations of Iron and Chlorine.*

As there are two oxides of iron, so there are also two distinct combinations of this metal and chlorine. One may be directly formed by the combustion of iron wire in chlorine gas; it is that volatile compound described by Sir HUMPHRY DAVY in his last Bakerian Lecture, which condenses after sublimation in the form of small brilliant iridescent plates. The other, I find, may be procured by heating to redness, in a glass tube with a very small orifice, the residue which is obtained by evaporating to dryness the green muriat of iron; it is a fixed substance requiring a red heat for its fusion; it is of a grayish but variegated colour, of a metallic splendour, and of a lamellar texture. As it absorbs chlorine when heated in this gas, and becomes entirely converted into the volatile compound, and as the volatile compound may likewise be obtained by heating in a glass tube nearly closed, the residue from the evaporation of the red muriat, it is evident that the

fixed compound contains less chlorine than the volatile, and that the former, consequently, may be called ferrane, and the latter ferranea.

Ferrane dissolves in water and forms the green muriat of iron; but the solution of the whole substance is not complete. There is always left a small and variable quantity of black oxide, which may be considered, on account of its variability, in a state of mechanical mixture, rather than of chemical union with the ferrane.

Ferranea is entirely soluble in water. The solution is identical with the red muriat of iron.

The analysis of both these compounds is easily effected by means of nitrat of silver.

50 grains of ferrane were put into water: the insoluble residue separated from the solution by decantation; washed, dried, and heated to redness for a minute, previously moistened with oil, weighed 3 grains, and was in the state of the black oxide, being attracted by the magnet. The solution entire, precipitated by nitrat of silver, afforded 102.5 grains of dried horn silver, which indicating 25.1125 grains of chlorine, the proportion of iron, omitting the 3 grains of oxide, appears to be 21.8875. And hence 100 of ferrane seem to consist of

53.43	chlorine
46.57	iron
100.00	

Ferranea is not easily obtained in considerable quantities, I have been obliged in consequence to operate upon small portions. The subject of analysis was procured by sublimation from the residue by evaporation of the red muriat. 20 grains

of this, in brilliant scales, were weighed in water. The solution, precipitated by nitrat of silver, yielded 53 grains of dried horn silver. Hence 100 of ferranea appear to consist of

64.9 chlorine
35.1 iron

100.0

4. *On the Combinations of Chlorine with Manganese, Lead, Zinc, Arsenic, Antimony, and Bismuth.*

I have attempted, by several methods, to obtain more than one combination of these different metals and chlorine, but without success.

I have procured a compound of manganese and chlorine, by evaporating to dryness the white muriat of this metal, and heating to redness the residue in a glass tube, having only a very small orifice. Muriatic acid vapour was produced, and a fixed compound remained, which required a red heat for its fusion, and was not altered by the strongest heat that could be given to it in the glass tube; but was rapidly decomposed when heated in an open vessel, muriatic acid fumes being evolved, and oxide of manganese formed, which was black or red, according to the intensity of the heat applied. The compound of manganese and chlorine is a very beautiful substance, it is of great brilliancy, generally of a pure delicate light pink colour and of a lamellar texture consisting of broad thin plates.

There is not much difficulty in obtaining this compound pure. Iron, with which manganese is commonly contaminated, may be separated by two or three repetitions of the solution of the compound in water, the evaporation to dryness of the

clear filtered muriat, and fusion of the residue procured by evaporation. Indeed, I think this a good general method for purifying manganese from iron. One of the combinations of the latter metal and chlorine being volatile, heat must separate it from the compound of manganese. And I have thus obtained it so free from iron, that triple prussiat of potash added to its solution in water, gave merely a white precipitate without the slightest tint of blue.

This compound deliquesces when exposed to the atmosphere, and is converted into the white muriat. Like ferrane, it affords a trifling residue when heated with water. The residue is oxide of manganese, white at first, but soon becoming red, and even black; it varies in quantity, according to the exclusion of air in the formation of the combination.

50 grains of the compound dissolved in water, with the exception of 1 grain; this residue was separated by decantation of the fluid, washed, dried, and heated to redness, it was in the state of black oxide. The colourless solution was precipitated by nitrat of silver. The horn silver formed, when dried, was equal to 108 grains. Hence, omitting the 1 grain of mixed oxide, 100 of this compound appear to consist of

54	chlorine
46	manganese
100	

The horn lead that I have analysed, was made by the decomposition of the nitrat of lead by muriatic acid, and it was well washed, dried, and fused in a glass tube with a small orifice. The strongest red heat that I could apply to it, under these circumstances, did not occasion its sublimation.

50 grains of it that had been fused were dissolved in water. This solution, heated with nitrat of silver, afforded 52.65 grains of dry horn silver. Hence 100 of horn lead appear to be composed of

25.78	chlorine
74.22	lead
100.00	

As this compound, when decomposed by an alkali, affords the protoxide of lead, it may be called plumbane.

The butter of zinc I have examined, was obtained by evaporating to dryness the muriat of this metal, and by heating to redness the residue in a glass tube. This compound is not volatile at a strong red heat in a close vessel, it fuses before it acquires a dull red heat, and on cooling it goes through several degrees of consistency, being viscid before it becomes solid.

This compound, when heated with water, affords a small residue of oxide of zinc, which, as in the preceding instances, may be considered as in the state of mechanical mixture.

In consequence of its powerful attraction for water, it is a very deliquescent substance; on this account it is necessary to weigh it in water to avoid error. 49.5 grains of it thus weighed, dissolved entirely in water, with the exception of 1 grain of oxide of zinc, which was separated by decantation and dried and ignited, and its quantity ascertained to be as stated. The solution precipitated by nitrat of silver afforded 99 grains of dried horn silver. Hence, excluding the 1 grain of oxide, 100 of butter of zinc seem to consist of

50 chlorine
50 zinc
———
100

This compound may be called zincane.

A compound of chlorine and arsenic has been long known, bearing the name of the fuming liquor of arsenic. It may be formed in several ways; by the combustion of arsenic in chlorine gas, by heating in a retort a mixture of arsenic and corrosive sublimate, or of arsenic and calomel, and by the distillation of muriat of arsenic with concentrated sulphuric acid. The old method by means of corrosive sublimate appears best adapted for procuring it in a pure state. About 6 parts of corrosive sublimate to 1 of arsenic are, I find, proper proportions. The mixture of the two substances should be intimate, and the heat applied to the retort for the distillation of the fuming liquor, gentle. When the liquor was not colourless at first, I have purified it by a second distillation.

The fuming liquor of arsenic, it is well known, is decomposed by water. The precipitate produced appears to be merely white oxide of arsenic, for, independent of other circumstances, it does not afford the fuming liquor when heated with strong sulphuric acid.

The fuming liquor, when gently heated, dissolves phosphorus, but it retains on cooling only a very small portion of this substance. The warm solution is not luminous in the dark.

The fuming liquor also, when warm, readily dissolves sulphur; indeed sulphur fused in the liquor seems capable of combining or of mixing with it in all proportions; but on cooling the greatest part of the sulphur is deposited, and assumes

a fine crystalline appearance; the form of the crystals was apparently the octahedron. This deposition seems to be merely sulphur with a little of the fuming liquor between the interstices of the crystals, for the crystals bear washing, and become tasteless superficially, but remain still acid internally, where the water has not penetrated.

It likewise dissolves resin. That which was called rosin was the subject of experiment. The solution was of a blueish green colour; but when gently heated it became brown, and remained so on cooling. The portion of resin the fuming liquor is capable of taking up, is very considerable; when the resin was added in excess, a viscid mixture was formed. The resinous solution was decomposed by water, and the resin was separated apparently unaltered mixed with white arsenic.

The fuming liquor is capable of combining with oil of turpentine and with olive oil. When the mixture was made with either of these oils, there was a considerable elevation of temperature, and a homogeneous colourless fluid was in each instance obtained.

In these and some other properties, the fuming liquor of arsenic is analogous to the fuming compounds of chlorine and sulphur, and chlorine and phosphorus; these too, having the power of dissolving sulphur, and phosphorus, and resin, and of entering into union with the fixed and volatile oils.

It is difficult to ascertain the proportion of the constituent parts of this compound by the ordinary modes of analysis, I have chosen therefore a synthetical method in preference; and from repeated experiments I find that 2 grains of arsenic require for complete conversion into the fuming liquor, 4 cubic inches exactly of chlorine gas.

The experiments were thus conducted: the arsenic in one piece was put into a small glass retort having a stop-cock, the retort was exhausted, and a known volume of chlorine gas was admitted from a graduated receiver by means of other stop-cocks, and the absorption of chlorine, after the entire conversion of the metal into the fuming liquor, was considered as the proportion condensed by the arsenic.

Now, since 100 cubic inches of chlorine gas weigh just 76.5 grains, 2 grains of arsenic combine with 3.06 grains of chlorine, the weight of 4 cubic inches of the gas. Hence 100 of the fuming liquor appear to consist of

60.48 chlorine
39.52 arsenic
―――
100.00

As the fuming liquor gives the white oxide when decomposed by water, arsenicane may be substituted for its old name.

The butter of antimony is a well known substance. That which I have examined was obtained by heating together corrosive sublimate and antimony, or antimony and calomel; and was always purified by a second distillation at a low temperature. The best proportion of corrosive sublimate and the metal for making the compound, I have found to be about $2\frac{1}{2}$ parts of the former to 1 part of the latter.

The butter of antimony, like arsenicane, is capable, when rendered fluid by heat, of dissolving resin and sulphur, and of combining with the fixed and volatile oils. It affects the oil of turpentine very like the liquor of Libavius; the action is considerable, much heat is produced, and the oil is rendered brown.

When the butter of antimony is decomposed by a sufficiently large quantity of the hydrosulphuret of potash, that compound is formed which is commonly called the golden sulphur of antimony, and which when decomposed by heat, I have found to afford merely water and sulphuret of antimony.*

To ascertain the proportion of antimony in the butter of antimony 60.5 grains of this substance colourless and crystallized, weighed in water, were heated in a solution of hydrosulphuret of potash. The whole of the antimony was dissolved, and the hydrosulphuret of potash being in excess, there was no precipitation on cooling. The solution was decomposed by muriatic acid, and the golden sulphur thus thrown down was collected on a filter well washed and dried; heated slowly to redness in a glass tube, steam in plenty was disengaged with very slight traces of sulphur, and sulphuret of antimony remained, which fused into one mass weighed 45 grains. According to the experiments of Proust, which I have repeated with the same result, sulphuret of antimony contains 74.1 per cent. of metal. Hence 45 grains of sulphuret or the 60.5 of butter of antimony, from which the sulphuret was procured, must contain 33.35 of metal; and considering the remainder 27.15 of the 60.5 as the proportion of chlorine, 100 of the

* These results appear to me to demonstrate the truth of M. Proust's opinion, that the golden sulphur is a hydrosulphuretted oxide of antimony. From my experiments the only difference of composition between kermes mineral and the preceding compound, seems to consist in the former containing a smaller proportion of sulphuretted hydrogene than the latter, for I have obtained by the decomposition of kermes mineral, by heat, a compound of sulphuret of antimony and protoxide, and I have converted kermes into the golden sulphur by means of water impregnated with sulphuretted hydrogene.

butter of antimony seem to consist of

39.58	chlorine
60.42	antimony
100.00	

This compound, as it yields when decomposed by water the submuriated protoxide, may be called antimoniane or stibiane.

A compound of bismuth and chlorine has been long known bearing the name of the butter of bismuth. It is obtained both when bismuth is heated with corrosive sublimate and calomel. 2 parts of corrosive sublimate to 1 part of metal, I have found good proportions for its preparation. There is some difficulty in procuring it pure and entirely free from the mercury revived; this is most readily effected by keeping the butter of bismuth in fusion, at a temperature just below that at which mercury boils; the mercury slowly subsides and collects in the bottom of the vessel, and this operation continued for an hour or two affords a pure or nearly pure butter of bismuth. Thus prepared, it is of a grayish white colour, opaque, uncrystallized, and of a granular texture. In a glass tube, with a very small orifice, it bears a red heat without subliming.

As a hydrosulphuret of bismuth is produced when the butter of bismuth is heated with the hydrosulphuret of potash, and as this hydrosulphuret, like that of antimony, affords, when decomposed by heat, a sulphuret and water, I have applied the same mode of analysis to this compound as to the last.

55 grains of butter of bismuth were decomposed in a warm solution of hydrosulphuret of potash. The dark brown hydrosulphuret of bismuth thus formed, and not dissolved, was collected on a filter; the hydrosulphuretted solution was

decomposed by muriatic acid, the slight precipitate of hydrosulphuret produced was added to the first portion, and the whole was well washed, dried, and heated to redness in a glass tube; the sulphuret of bismuth thus obtained, fused into one mass, weighed 44.7 grains. I had previously ascertained the proportion of metal in this sulphuret, and found it to be 81.8 per cent. 44.7 grains of sulphuret, or 55 grains of the butter, must therefore contain 36.5 grains of bismuth; and hence, 100 of bismuth appear to consist of

33.6 chlorine
66.4 bismuth

100.0

The butter of bismuth may be called bismuthane.

Among the preceding combinations of the metals and chlorine, there is a surprising difference in respect to volatility and fusibility. Iron and manganese, two difficultly fusible metals, form with chlorine readily fusible compounds, and a combination of the former metal and chlorine is even volatile; the compounds of tin and chlorine, and of chlorine and antimony, are very volatile substances, though the metals themselves are fixed at very high temperatures; on the contrary, the combinations of chlorine with bismuth, zinc, and lead, do not exceed in fusibility; indeed are not quite so fusible as the metals themselves. I can offer no explanation of these phenomena.

Another singularity attending the liquid fuming compounds of chlorine, such as the liquor of Libavius, the fuming liquor of arsenic, and the oxymuriats of sulphur and phosphorus, is, that they do not become solid at low temperatures. I have

reduced, by means of a mixture of snow and muriat of lime, the temperature of all these substances 20 degrees below the zero of FAHRENHEIT's thermometer, but without affecting their liquidity.

The influence of atmospheric air on the compounds of the metals and chlorine at high temperatures is curious, and worthy of particular attention. The combinations of chlorine with lead, zinc, copper, and bismuth, appear to be volatile in open vessels, and fixed in closed ones. How moist air operates in these instances, it is difficult to say. In other cases, where it evidently acts chemically, the changes explain themselves; thus, when the compounds of iron and chlorine and of manganese and chlorine are heated in the open air, hygrometrical water of the atmosphere seems to be decomposed, as muriatic acid fumes are produced, and oxides of the metals formed. Probably the volatility of the other compounds is connected with similar circumstances. This action of moist air has hitherto been much neglected; it is certainly worthy of being more fully inquired into, both in a theoretical and practical point of view. Its importance in practice is exemplified in the reduction of horn silver, and in the formation of several of the compounds of chlorine and the metals; if moist air be admitted in these operations, the silver will be lost, and the compounds not formed.

Guided by analogy, I have been led to try whether the muriat of magnesia, which is readily decomposed by heat in the open air, would not, when the air was excluded, by introducing it into a glass tube with a very small orifice, afford a permanent compound. The result was agreeable to my expectations; I obtained, by strongly heating the muriat for a

quarter of an hour, a substance like enamel in appearance, being semi-fused, and which appeared to be a mixture of magnesia and the true compound of magnesium and chlorine, for heated with water magnesia was separated, and a muriat of magnesia formed.

5. *On the Relation between the Proportion of Oxygene and Chlorine in Combination with several Metals.*

Errors being very common in chemical analyses, even in those conducted most skilfully and carefully, all possible means should be taken to discover them; and no means, I think, promise to be more effectual for this purpose, than the general analogy of definite proportions. From a great variety of facts, it appears that oxygene and chlorine combine with bodies in the ratio of 7.5 to 33.6. With 1 part by weight of hydrogene, for example, 7.5 of oxygene unite to form water, and 33.6 of chlorine unite with the same proportion to produce muriatic acid gas. To judge therefore of the accuracy of the analyses of the preceding combinations of the metals and chlorine, it is only necessary to compare them with the analyses of the oxides of the same metals. If the two agree, there will be reason to consider them both correct, but should they disagree, there is equal reason for supposing one or both of them to be wrong.

Thus, as the orange oxide of copper is analogous to cuprane and the brown oxide to cupranea, the oxygene and chlorine should be to each other in these compounds as 7.5 to 33.6. And from comparison of my analysis, with those of Mr. CHENEVIX and M. PROUST, it appears, that in the two first, copper being as 60, the oxygene is to the chlorine as 7.79, instead of 7.5 to

33.77, instead of 33.6; and in the two last as 7.5 to 33.6, or as 15 to 67.2. Coincidences as near as might be reasonably

There is not the same agreement between M. PROUST's analyses of the oxides of tin and the preceding ones of the combinations of this metal and chlorine. This discordance induced me to repeat my analyses, and obtaining the same result as at first, I directed my attention to the oxides of tin, and made the following experiments to ascertain the proportion of their constituent parts.

42.5 grains of tin, which had been precipitated from the muriat of this metal by zinc, were heated with nitric acid in a platina crucible, and slowly converted into peroxide; the acid and water were driven off by gentle evaporation at first, and afterwards by a strong red heat continued for a quarter of an hour. The peroxide thus produced was of a light yellow colour, and being very gradually dried, it was semi-transparent, and hard enough to scratch glass; it weighed 54.25 grains. Hence, as 42.5 grains of tin acquire, on conversion into peroxide, 11.75 grains of oxygene, this oxide appears to contain 21.66 per cent. of oxygene, just the quantity found in the native oxide by KLAPROTH, instead of 28, the proportion stated by PROUST.

M. BERTHOLLET, jun. has shewn that M. PROUST's estimate of 20 per cent. of oxygene in the protoxide is incorrect. To ascertain the true proportion, 20 grains of tin were dissolved in strong muriatic acid in a retort connected with a pneumatic apparatus, and without the assistance of heat; 16 cubic inches of hydrogene gas were produced. (Barom. 30, thermom. 60) as the production of this quantity of hydrogene indicates an

absorption of oxygene by the tin equivalent to 8 cubic inches, or (as 100 cubic inches weigh 34.2 grains) to 2.736 grains, the protoxide of tin appears to contain 11.99 per cent. of oxygene.

These analyses of the oxides, compared with those of the combinations of tin and chlorine, are found very nearly to agree. The ratio of oxygene to chlorine in the two first similar compounds, the tin being as 55, is as 7.5 to 33.4; and in the two last, viz. the peroxide and the liquor of Libavius, as 7.6 to 33.5, or as 15.2 to 67.

As the black oxide of iron is formed by the decomposition of ferrane by a solution of potash, and the red oxide by that of ferranea, it is evident that these oxides and combinations of iron and chlorine should coincide in the proportions of their constituent parts. This appears from the analyses* of Dr. THOMPSON to be nearly the case, for iron being as 29.5, the oxygene is to the chlorine in the black oxide and ferrane as 8 instead of 7.5 to 33.6; and in the two others as 8 to 33.6, or as 13.2 to 55.5. Here the agreement is less than in other instances; but this is not surprising considering the different estimates of the proportions of oxygene in the oxides of iron, and the difficulty of ascertaining them correctly.

The yellow oxide of lead and the white oxides of antimony, bismuth, zinc, and arsenic are formed, when the combinations of these metals and chlorine are decomposed by a solution of potash. But on comparison with the best analyses of the oxides, there is not, excepting in the case of zinc and arsenic, that coincidence of proportions which might be expected. Zinc being as 34.5, the oxygene in the oxide from the analysis

* NICHOLSON's Journal, Vol. XXVII. p. 375.

of PROUST, is to the chlorine as 7.5 to 34.4; and the arsenic being as 21.9, the oxygene, from the analysis of the same chemist, is to the chlorine as 7.3 to 33.6. The analyses of the oxides of the other metals being at variance with those of the chlorine combinations, I was induced to make the following experiments, with the hope of discovering the cause of the difference.

100 grains of lead, which had been precipitated from the nitrat of lead by zinc, were dissolved in nitric acid and thrown down by carbonat of potash. This precipitate of carbonat of lead was well washed and dried and heated to dull redness for a quarter of an hour in a platina crucible; by this treatment all the carbonic acid was expelled; the remaining yellow oxide weighed 107.7 grains, and it dissolved in muriatic acid without effervescing, and without affording any residue of brown oxide. Hence the yellow oxide of lead appears to contain 7.15 per cent. of oxygene. And this proportion of oxygene in the oxide compared with that of chlorine in plumbane, lead being as 97.2 appears to be in the ratio of 7.5 to 33.8, instead of that of 15.6 the estimate of KLAPROTH, or of 11.2 the estimate of Dr. THOMPSON to 33.8. KLAPROTH might have been misled by considering the hydrated oxide as a true white oxide free from water.

According to M. PROUST the peroxide of antimony contains 23 per cent. of oxygene, and the protoxide 18.* I have repeated this chemist's experiments; my results, in which the peroxide is concerned, agree with his; but there is not the same concordance in those relating to the protoxide. The protoxide I used was either prepared by the decomposition of

* Journal de Physique, Tom. LV.

the butter of antimony, or of the sulphat, by a boiling solution of carbonat of potash. This oxide, in its purest state, I have always found as M. Proust describes it, of a light fawn colour before fusion, and afterwards in mass of a gray colour, and of a radiated crystalline texture. 100 grains of it that had been fused were heated in the state of powder with strong test nitric acid in a platina crucible, when nitrous gas ceased to be produced, the excess of nitric acid was expelled by a gentle heat, and the oxide was heated to dull redness, the increase of weight after this, was equal to 10.4 grains; nitric acid was again added and the process repeated, but without any alteration of weight being produced. Hence as the peroxide contains 23 per cent. the protoxide seems to contain 15 per cent.; which proportion of oxygene very nearly agrees with that of chlorine in the butter of antimony, for antimony being as 42.5, the former is to the latter as 7.5 to 34.6, instead of 33.6. I put some confidence in this estimate of the proportion of oxygene in the protoxide, not only on account of its agreement with the analysis of the butter of antimony, but because it was confirmed on the repetition of the experiment.

Klaproth concludes from his experiments, that the oxide of bismuth, prepared by means of nitric acid, contains 17.7 per cent. of oxygene, and in consequence this oxide has been considered distinct from that which is formed by direct calcination of the metal, and which contains a much smaller proportion. But there is reason to believe that this difference does not really exist, and that there is only one known oxide of bismuth, and that Klaproth's oxide was an hydrated oxide; for I have found that 100 grains of bismuth, converted by nitric acid into oxide, precisely in the same manner as the

protoxide of antimony was more highly oxidated, gained only 11.1 grains. Klaproth did not heat his oxide to redness, and hence apparently the discordance. From the above result, which I have confirmed by repetition of the experiment, oxide of bismuth seems to contain 10 per cent. of oxygene and bismuth being as 67.5, the oxygene in the oxide is to the chlorine in the butter of bismuth, as 7.5 to 34.2.

6. On the Relation between the Proportion of Sulphur in the Sulphurets, and the Proportion of Chlorine in some of the Combinations of Chlorine and the Metals.

The last section afforded proofs of the useful application of the general analogy of definite proportions in correcting the results of chemical analyses. In the present section, it is my intention to pursue a little further, the plan that I have adopted in the preceding, and to apply another test to the analyses of the combinations of the metals and chlorine, by comparing some of them with the combinations of the same metals and sulphur.

I was first led to examine the sulphurets of tin on a different account. Aurum musivum, it has been observed, is formed when stannane is heated with sulphur. According to M. Proust, this substance is a sulphuretted oxide of tin. Were this opinion correct, an argument might evidently be deduced from it, in favour of the existence of oxygene in chlorine. To satisfy myself respecting this, I endeavoured to ascertain whether any sulphureous acid gas is produced by the decomposition of aurum musivum by heat, as it is commonly asserted. I heated to redness in a bent luted green glass tube connected with a pneumatic mercurial apparatus about 20 grains of

aurum musivum, prepared by the decomposition of stannane with sulphur, no more gas was produced than the expansion by heat occasioned, sulphur sublimed, and a gray sulphuret of tin remained. These results I have several times obtained, and not only with aurum musivum prepared as the preceding, but with some also made according to WOULFE's process. As no sulphureous acid gas was produced, and as sulphur sublimed, it may be concluded that aurum musivum differs merely from the gray sulphuret in containing a larger quantity of sulphur. My next object was to ascertain the exact proportion of sulphur in both these sulphurets, for the sake of comparison with the combinations of tin and chlorine.

100 grains of tin in a finely divided state, as precipitated from the muriat of this metal by zinc, were heated in a glass tube intimately mixed with sulphur, the combination of the two was accompanied with vivid ignition, the sulphuret formed weighed 127.3 grains, and broken, it appeared perfectly homogeneous; it was pounded, and again heated with sulphur; but the excess of sulphur being expelled, the fused sulphuret had not increased in weight. The second time I made this experiment, I obtained the same result.

50 grains of aurum musivum, purified from mixed sulphur by exposure in a close vessel to a dull red heat, were decomposed by a bright red heat in a small green glass tube nicely weighed, and having only a very small orifice; the loss of sulphur, by conversion into the gray sulphuret, was equal to 9.3 grains. Hence, as 40.7 grains of gray sulphuret contain 8.72 grains of sulphur, 50 grains of aurum musivum appear to contain 18.02 grains.

The ratio in which sulphur combines with bodies is to that

in which oxygene and in which chlorine combines, as 15 to 7.5 and 33.6. This appears from the proportions of the constituent parts of sulphuretted hydrogene and sulphureous acid gas, for I have found 100 cubic inches of the former to weigh 36.64 grains, and 100 of the latter 68.44 grains. In the comparison, therefore, between the sulphurets of tin and the combinations of this metal and chlorine, 15 by weight of sulphur are equivalent to 33.6 of chlorine. And the tin being as 55, it appears from the analysis of the gray sulphuret and stannane, that the sulphur is to the chlorine as 15 exactly to 33.4; and from the analysis of the other two compounds, aurum musivum and the liquor of Libavius, as 15.5 to 33.5, or as 31 to 67.

The proportions of sulphur in the two sulphurets of iron, do not accord with the proportions of oxygene in the oxides, or of chlorine in the chlorine combinations; but I am yet ignorant of the cause of this difference.

100 grains of lead, heated with sulphur in a glass tube, afforded, in two trials, 115.5 grains of fused sulphuret. Hence lead being as 97.2, the sulphur is to the chlorine in the respective combinations as 15.09 to 33.8.

Sulphuret of antimony contains 25.9 per cent. of sulphur. Hence antimony being as 42.5, the sulphur in the sulphuret is to the chlorine in the butter of antimony, as 14.86 to 34.6.

100 grains of bismuth heated with sulphur afforded 122.3 grains of sulphuret. Hence bismuth being as 67.5, the sulphur is to the chlorine as 15.08 to 34.2.

In the following table, the proportions are collected in which chlorine, sulphur, and oxygene combine with several metals; the numbers representing the metals are kept constantly the same, for the greater facility of comparison.

Copper	60	+ 32.77	chlorine	= cuprane.
		+ 67.20	ditto	= cupranea.
		+ 7.79	oxygene	= orange oxide.
		+ 15.00	ditto	= brown oxide.
Tin	55	+ 33.40	chlorine	= stannane.
		+ 67.00	ditto	= stannanea.
		+ 15.00	sulphur	= gray sulphuret.
		+ 31.00	ditto	= aurum musivum.
		+ 7.50	oxygene	= protoxide.
		+ 15.20	ditto	= peroxide.
Iron	29.5	+ 33.60	chlorine	= ferrane.
		+ 55.50	ditto	= ferranea.
		+ 8.00	oxygene	= black oxide.
		+ 13.20	ditto	= red oxide.
Manganese	28.4	+ 33.60	chlorine.	
Lead	97.2	+ 33.80	chlorine	= plumbane.
		+ 15.09	sulphur	= sulphuret.
		+ 7.50	oxygene	= yellow oxide.
Zinc	34.5	+ 34.40	chlorine	= zincane.
		+ 7.50	oxygene	= oxide.
Arsenic	21.9	+ 33.60	chlorine	= arsenicane.
		+ 7.30	oxygene	= white oxide.
Antimony	42.5	+ 34.60	chlorine	= antimonane.
		+ 14.86	sulphur	= sulphuret.
		+ 7.50	oxygene	= protoxide.
Bismuth	67.5	+ 34.20	chlorine	= bismuthane.
		+ 15.08	sulphur	= sulphuret.
		+ 7.50	oxygene	= oxide.

7. *On the Action of muriatic Acid on some Combinations of Chlorine and Metals.*

Sir HUMPHRY DAVY has pointed out in a great variety of instances, the existence of an analogy between chlorine and oxygene. He has shewn that the former, united with certain inflammables, constitutes, like the latter, acid compounds; and combined with metals, as it has already been observed, substances similar in many respects to metallic oxides.

I have kept this analogy in view in my inquiries, and directed by it in my experiments, I have obtained some results which appear to me to coincide with it.

Thus having been led to try the action of muriatic acid on different combinations of the metals and chlorine, I have found many of them capable of uniting with this acid, and of forming compounds not dissimilar to some of those consisting of acids and metallic oxides.

Corrosive sublimate, stannane, cuprane, and the combinations of chlorine with antimony, zinc, lead, and silver are all soluble in different degrees in muriatic acid.

Corrosive sublimate, which is but sparingly soluble in water, and still more sparingly in the sulphuric and nitric acids, is, I have ascertained, very readily soluble in muriatic acid. 1 cubic inch of the common strong acid takes up about 150 grains of this substance, and when gently heated, a quantity far more considerable, about 1000 grains. The compound thus formed solidifies on cooling into a crystalline fibrous mass of a pearly and brilliant lustre. It is decomposed by heat, the acid being first expelled, and when exposed to the atmosphere, it efflo-

resces and appears to lose its acid, for afterwards analysed, it is found to be pure corrosive sublimate.

When I first tried the action of muriatic acid on the different combinations of chlorine already mentioned, I was not aware that KLAPROTH had before observed the solubility of horn silver in this acid, and Mr. CHENEVIX that of cuprane. Horn silver, cuprane, and horn lead are precipitated from muriatic acid, unaltered by water. Both the hot saturated solutions of the two last compounds deposit crystals on cooling; those, from the solution of the former, are of an olive green colour and of a prismatic form, and consist of cuprane and muriatic acid; those from the latter, are small white brilliant plates.

Finding the combinations of the metals and chlorine, so generally soluble in liquid muriatic acid, I expected that some of them might absorb muriatic acid gas; but none that I have tried have possessed this property, not even the liquor of Libavius. Indeed this is not singular, for water is necessary to the composition of many saline bodies, neutral carbonat of ammonia and nitrat of ammonia, for instance, cannot be formed without the presence of water. Neither is the precipitation of cuprane, horn silver, and horn lead from muriatic acid by water extraordinary; there are several salts containing metallic oxides which are liable to the same change, the oxides having less affinity for the acid, than water has.

The action of muriatic acid on the combinations of the different metals and chlorine will, I have little doubt, afford, when more minutely investigated, explanations of many phenomena which are not yet well accounted for. Before I conclude, I shall mention only one instance to which it already appears

to be applicable. M. PROUST has observed the decomposition of calomel by boiling muriatic acid, and its conversion into corrosive sublimate and running mercury. Now calomel being insoluble in muriatic acid, these changes evidently appear to be owing to the strong attraction of the acid, for corrosive sublimate, which has been already shewn to exist.

XI. *Further Experiments and Observations on the Action of Poisons on the Animal System.* *By* B. C. Brodie, *Esq. F. R. S.* *Communicated to the Society for the Improvement of Animal Chemistry, and by them to the Royal Society.*

Read February 27, 1812.

I.

Since I had the honour of communicating to the Royal Society some observations on the action of certain poisons on the animal system, I have been engaged in the further prosecution of this inquiry. Besides some additional experiments on vegetable poisons, I have instituted several with a view to explain the effects of some of the more powerful poisons of the mineral kingdom. The former correspond in their results so nearly with those which are already before the public, that, in the present communication, I shall confine myself to those which appear to be of some importance, as they more particularly confirm my former conclusions respecting the recovery of animals apparently dead, where the cause of death operates exclusively on the nervous system. In my experiments on mineral poisons, I have found some circumstances wherein their effects differ from those of vegetable poisons, and of these I shall give a more particular account. Whatever may be the value of the observations themselves, the subject must be allowed to be one that is deserving of investigation, as it does not appear unreasonable to expect that such investigation may hereafter lead to some improvements

in the healing art. This consideration, I should hope, will be regarded as a sufficient apology for my pursuing a mode of inquiry by means of experiments on brute animals, of which we might well question the propriety, if no other purpose were to be answered by it than the gratification of curiosity.

In my former communication on this subject, I entered into a detailed account of the majority of my experiments. This I conceived necessary, because in the outset of the inquiry I had been led to expect that even the same poison might not always operate precisely in the same manner; but I have since had abundant proof, that in essential circumstances there is but little variety in the effects produced by poisons of any description, when employed on animals of the same, or even of different species, beyond what may be referred to the difference in the quantity, or mode of application of the poison, or of the age and power of the animal. This will explain the reason of my not detailing, in the present communication, so many of the individual experiments from which my conclusions are drawn, as in the former; at the same time I have not been less careful to avoid drawing general conclusions from only a limited number of facts. Should these conclusions prove fewer, and of less importance than might be expected, such defects will, I trust, be regarded with indulgence; at least by those, who are aware of the difficulty of conducting a series of physiological experiments; of the time, which they necessarily occupy; of the numerous sources of fallacy and failure which exist; and of the laborious attention to the minutest circumstances, which is in consequence necessary in order to avoid being led into error.

II. *Experiments with the Woorara.*

In a former experiment, I succeeded in recovering an animal, which was apparently dead from the influence of the essential oil of bitter almonds, by continuing respiration artificially until the impression of the poison upon the brain had ceased; but a similar experiment on an animal under the influence of the woorara was not attended with the same success. Some circumstances led me to believe, that the result of the experiment with the woorara might have been different, if it had been made with certain precautions; but I was unable at that time to repeat it, in consequence of my stock of the poison being exhausted. I have since, however, been able to procure a fresh supply, and I shall relate two experiments which I have made with it. In one of these, an animal apparently dead from the woorara, was made to recover, notwithstanding the functions of the brain appeared to be wholly suspended for a very long period of time; in the other, though ultimate recovery did not take place, the circulation was maintained for several hours after the brain had ceased to perform its office.

Experiment 1. Some woorara was inserted into a wound in a young cat. She became affected by it in a few minutes, and lay in a drowsy and half sensible state, in which she continued at the end of an hour and fifteen minutes, when the application of the poison was repeated. In four minutes after the second application, respiration entirely ceased, and the animal appeared to be dead; but the heart was still felt acting about one hundred and forty times in a minute. She was placed in

a temperature of 85 of FAHRENHEIT's thermometer, and the lungs were artificially inflated about forty times in a minute.

The heart continued acting regularly.

When the artificial respiration had been kept up for forty minutes, the pupils of the eyes were observed to contract and dilate on the increase or diminution of light; saliva had flowed from the mouth, and a small quantity of tears was collected between the eye and eye-lids; but the animal continued perfectly motionless and insensible.

At the end of an hour and forty minutes, from the same period, there were slight involuntary contractions of the muscles, and every now and then there was an effort to breathe. The involuntary motions continued, and the efforts to breathe became more frequent. At the end of another hour, the animal, for the first time, gave some signs of sensibility when roused, and made spontaneous efforts to breathe twenty-two times in a minute. The artificial respiration was discontinued. She lay, as if in a state of profound sleep, for forty minutes, when she suddenly awoke, and walked away. On the following day she appeared slightly indisposed; but she gradually recovered, and is at this time still alive and in health.

Experiment 2. Some woorara was applied to a wound in a rabbit. The animal was apparently dead in four minutes after the application of the poison; but the heart continued acting. He was placed in a temperature of 90°, and the lungs were artificially inflated. The heart continued to act about one hundred and fifty times in a minute. For more than three hours the pulse was strong and regular; after this, it became feeble and irregular, and at the end of another hour the circulation had

entirely ceased. During this time there was no appearance of returning sensibility.

The circulation of the blood may be maintained in an animal from whom the brain has been removed for a considerable, but not for an unlimited time. We may conclude, that in the last of these experiments the animal did not recover, because the influence of the poison continued beyond the time during which the circulation may be maintained without the brain.

III. *On the Effects of Arsenic.*

When an animal is killed by arsenic taken internally, the stomach is found bearing marks of inflammation; and it is a very general opinion, 1, that this inflammation is the cause of death: 2, that it is the consequence of the actual contact of the arsenic with the internal coat of the stomach. But in several cases I have found the inflammation of the stomach so slight, that on a superficial examination it might have been easily overlooked; and in most of my experiments with this poison death has taken place in too short a time for it to be considered as the result of inflammation: and hence we may conclude, that the first of these opinions is incorrect; at least as a general proposition.

Many circumstances conspire to show that the second of these opinions also is unfounded.

In whatever way the poison is administered, the inflammation is confined to the stomach and intestines; I have never seen any appearance of it in the pharynx or œsophagus.

Mr. Home informed me, that in an experiment made by Mr. Hunter and himself, in which arsenic was applied to a

wound in a dog, the animal died in twenty-fours, and the stomach was found to be considerably inflamed.

I repeated this experiment several times, taking the precaution always of applying a bandage to prevent the animal licking the wound. The result was, that the inflammation of the stomach was commonly more violent and more immediate, than when the poison was administered internally, and that it preceded any appearance of inflammation of the wound.* Some experiments are already before the public, which led me to conclude that vegetable poisons, when applied to wounded surfaces, affect the system by passing into the circulation through the divided veins. From this analogy, and from all the circumstances just mentioned, it may be inferred that arsenic, in whatever way it is administered, does not produce its effects even on the stomach until it is carried into the blood.

But the blood is not necessary to life, except so far as a constant supply of it is necessary for the maintenance of the functions of the vital organs. The next object of inquiry therefore is, when arsenic has entered the circulation, on what organs does it operate, so as to occasion death?

When arsenic is applied to an ulcerated surface, it produces a slough, not by acting chemically, like caustics in general, but by destroying the vitality of the part to which it is applied,

* Since the greater part of my experiments on this subject were made, I have seen an account of an inaugural Dissertation on the Effects of Arsenic, by Dr. Jaëger of Stuttgard. Dr. Jaëger has come to conclusions similar to those above stated, that in an animal killed by arsenic, the inflammation of the stomach is not the cause of death, and that the poison does not produce its fatal effects until it has entered the circulation. I have to regret that I have had no opportunity of seeing the original of this Dissertation.

independently of chemical action. This led me at first to suppose, that when arsenic has passed into the circulation, death is the consequence, not so much of the poison disturbing the functions of any particular organ, as of its destroying at once the vitality of every part of the system. The following circumstances, however, seem to show that this opinion is erroneous. In an animal under the full influence of arsenic, even to the instant of death, some of the secretions, as those of the kidneys, stomach, and intestines, continue to take place in large quantity; and the muscles are capable of being excited, after death, to distinct and powerful contractions by means of the Voltaic battery.

Experiment 3. Seven grains of the white oxide of arsenic were applied to a wound in the back of a rabbit.

In a few minutes he was languid, and the respirations were small and frequent. The pulse was feeble, and after a little time could not be felt. The hind legs became paralysed.* He grew insensible, and lay motionless, but with occasional convulsions. At the end of fifty-three minutes from the time of the arsenic being applied, he was apparently dead; but on opening the thorax, the heart was found still acting, though very slowly and feebly. A tube was introduced into the trachea, and the lungs were artificially inflated; but this appeared

* I have observed, that where the functions of the brain are disturbed, paralysis first takes place in the muscles of the hind legs; afterwards in those of the trunk and fore legs; and last of all in the muscles of the ears and face. These facts seem to show that the influence of the brain, like that of the heart, is not propagated with the same facility to the distant as to the near organs; and this is further confirmed by cases of disease which occasionally occur, in which, although the paralysis is confined to the lower half of the body, the morbid appearances met with on dissection are entirely confined to the brain.

to have no effect in prolonging the heart's action. On dissection, the inner membrane of the stomach was found slightly inflamed.

Experiment 4. Two drams of arsenic acid dissolved in six ounces of water were injected into the stomach of a dog, by means of a tube of elastic gum, passed down the œsophagus. In three minutes he vomited a small quantity of mucus, and this occurred again several times. The pulse became less frequent, and occasionally intermitted. At the end of thirty-five minutes the hind legs were paralysed, and he lay in a half sensible state. At the end of forty-five minutes he was less sensible; the pupils of the eyes were dilated; the pulse had fallen from 140 to 70 in a minute, and the intermissions were frequent. After this, he became quite insensible; convulsions took place, and at the end of fifty minutes, from the beginning of the experiment, he died. On opening the thorax, immediately after death, tremulous contractions of the heart were observed; but not sufficient to maintain the circulation. The stomach and intestines contained a large quantity of mucous fluid, and their internal membrane was highly inflamed.

These experiments were repeated, and the results, in all essential circumstances, were the same. The symptoms produced were, 1, paralysis of the hind legs, and afterwards of the other parts of the body; convulsions; dilatation of the pupils of the eyes; insensibility; all of which indicate disturbance of the functions of the brain: 2, a feeble, slow, intermitting pulse, indicating disturbance of the functions of the heart. Where the heart has continued to act after apparent death, I have never, in any one instance, been able to prolong its action by means of artificial respiration. 3, pain in

the region of the abdomen; preternatural secretion of mucus from the alimentary canal; sickness and vomiting in those animals, which are capable of vomiting; symptoms which arise from the action of the poison on the stomach and intestines. There is no difference in the effects of arsenic, whether it is employed in the form of white oxide, or of arsenic acid, except that the latter is a more active preparation. When arsenic is applied to a wound, the symptoms take place sooner than when it is given internally; but their nature is the same.

The symptoms produced by arsenic may be referred to the influence of the poison on the nervous system, the heart,* and the alimentary canal. As of these the two former only are concerned in those functions, which are directly necessary to life, and as the alimentary canal is often affected only in a slight degree, we must consider the affection of the heart and nervous system as being the immediate cause of death.

In every experiment which I have made with arsenic, there were evident marks of the influence of the poison on all the organs which have been mentioned; but they were not in all cases affected in the same relative degree. In the dog, the affection of the heart appeared to predominate over that of the

* When I say that a poison acts on the heart, I do not mean to imply that it necessarily must act directly on the muscular fibres of that organ. It is highly probable, that the heart is affected only through the medium of its nerves; but the affection of the heart is so far independent of the affection of the nervous system generally, that the circulation may cease although the functions of the brain are not suspended, and the functions of the brain may be wholly suspended without the circulation being at all disturbed. In proof of the first of these propositions, I may refer to my former experiments on the upas antiar, in which the sensibility of the animal continued to the very instant of death; and respiration, which is under the influence of the brain, continued even after the heart had ceased to act. In proof of the second, I may refer, among many others, to the experiments detailed in the Croonian Lecture for 1810.

brain, and on examining the thorax, immediately after death, this organ was found to have ceased acting, and in a distended state. In the rabbit, the affection of the brain appeared to predominate over that of the heart, and the latter was usually found acting slowly and feebly, after the functions of the brain had entirely ceased. In the rabbit, the effects of the arsenic on the stomach and intestines were usually less than in carnivorous animals.

The action of arsenic on the system is less simple than that of the majority of vegetable poisons. As it acts on different organs, it occasions different orders of symptoms, and as the affection of one or another organ predominates, so there is some variety in the symptoms produced even in individual animals of the same species.

In animals killed by arsenic the blood is usually found fluid in the heart and vessels after death; but otherwise all the morbid appearances met with on dissection are confined to the stomach and intestines. As this is the case, and as the affection of these organs occasions remarkable symptoms, it may be right to mention the result of my observations on this subject.

In many cases where death takes place, there is only a very slight degree of inflammation of the alimentary canal: in other cases the inflammation is considerable. It generally begins very soon after the poison is administered, and appears greater or less according to the time which elapses before the animal dies. Under the same circumstances, it is less in graminivorous than in carnivorous animals. The inflammation is greatest in the stomach and intestines; but it usually extends also over the whole intestine. I have never observed inflam-

mation of the œsophagus. The inflammation is greater in degree, and more speedy in taking place, when arsenic is applied to a wound, than when it is taken into the stomach. The inflamed parts are in general universally red, at other times they are red only in spots. The principal vessels leading to the stomach and intestines are turgid with blood; but the inflammation is usually confined to the mucous membrane of these viscera, which assumes a florid red colour, becomes soft and pulpy, and is separable without much difficulty from the cellular coat, which has its natural appearance. In some instances there are small spots of extravasated blood on the inner surface of the mucous membrane, or between it and the cellular coat, and this occurs independently of vomiting. I have never, in any of my experiments, found ulceration or sloughing of the stomach or intestine; but if the animal survives for a certain length of time, after the inflammation has begun, it is reasonable to conclude that it may terminate in one or other of these ways.

I am diposed to believe that sloughing is very seldom, if ever, the direct consequence of the application of arsenic to the stomach or intestines. Arsenic applied to an ulcer will occasion a slough; but its action in doing this is very slow. When I have applied the white oxide of arsenic to a wound, though the animal has sometimes lived three or four hours afterwards, and though violent inflammation has taken place in the stomach and intestines, I have never seen any preternatural appearance in the part to which it was applied, except a slight effusion of serum into the cellular membrane. Arsenic speedily produces a very copious secretion of mucus and watery fluid from the stomach and intestines, which separates it from actual

contact with the inner surface of these organs, even though taken in large quantity and in substance; and in animals which are capable of vomiting, by much the greater part is rejected from the stomach very soon after it has been taken in. Hence, though a few particles of arsenic are sometimes found entangled in the mucus, or in the coagulum of extravasated blood, and adhering to the inner surface of the stomach, I have never seen it in such a quantity as might be supposed capable of producing a slough. In one instance, where a dog had swallowed a large quantity arsenic in substance, a brown spot, about an inch in diameter, was observed after death on the inner surface of the cardiac extremity of the stomach, having so much of the appearance of a slough, that at first I had no doubt of it being so; but on examination this proved to be only a thin layer of dark coloured coagulum of blood, adhering very firmly to the surface of the mucous membrane, and having a few particles of arsenic entangled in it. On removing this the mucous membrane still appeared of a dark colour; but this was also found to arise from a thin layer of coagulum of blood between it and the cellular coat. The mucous membrane itself was inflamed; but otherwise in a natural state. I have observed a similar appearance, but occupying a less extent of surface, several times. In the Hunterian Museum there is a human stomach, which was preserved to show what was considered as a slough produced by the action of arsenic. On examining this preparation, I found that the dark coloured spot, which had been supposed to be a slough, was precisely of the same nature with that just described.

Although the affection of the stomach and intestines from arsenic is not the cause of death, under ordinary circumstances,

it is reasonable to conclude that it may be so in some instances, if the animal survives the effects produced on the organs more immediately necessary to life. Mr. HENRY EARLE informed me of an instance, in which this appeared to be the case. A woman in St. Bartholomew's hospital, who had taken arsenic, recovered of the immediate symptoms, but died at the end of four or five days. On examination after death, extensive ulcerations were found of the mucous membrane of the stomach and intestines, which we can hardly doubt to have been the cause of death.

It is an important matter of inquiry, as connected with judicial medicine, how far may the examination of the body, after death, enable us to decide, whether an animal has died of the effects of arsenic? On this subject, however, I have only a few remarks to make.

The inflammation from arsenic, occupying in general the whole of the stomach and intestine, is more extensive than that from any other poison with which I am acquainted. It does not affect the pharynx or œsophagus, and this circumstance distinguishes it from the inflammation which is occasioned by the actual contact of irritating applications.

But little in general is to be learnt from the examination of the contents of the stomach after death. When arsenic has been taken in substance, small particles of it are frequently found entangled in the mucus, or in the extravasated blood; but where this was not the case, I have never known, in an animal that was capable of vomiting, that arsenic could be detected in the contents of the stomach after death, though examined by the most accurate chemical tests. As some substances when taken internally are separated from the blood

very soon afterwards with the urine, I thought it probable that arsenic might be separated with the urine also; but Mr. BRANDE (to whom I am indebted for assistance on this, as well as on many other occasions) could never detect the smallest trace of arsenic in it.

IV. *Experiments with the Muriate of Barytes.*

When barytes is taken into the stomach, or applied to a wound, it is capable of destroying life; but when in its uncombined state its action is very slow. The muriate of barytes, which is much more soluble than the pure earth, is (probably on this account) a much more active poison.

Experiment 5. Ten grains of muriate of barytes rubbed very fine, and moistened with two drops of water, were applied to two wounds in the thigh and side of a rabbit. In four minutes he was evidently under the influence of the poison. In a short time he became giddy: then his hind legs were paralysed; and he gradually fell into a state of insensibility, with dilated pupils, and lay, in general motionless, but with occasional convulsions. The pulse beat 150 in a minute, but feeble, and it occasionally intermitted. He was apparently dead in twenty minutes from the application of the poison; but on opening the chest, the heart was found still acting, and nearly three minutes elapsed before its action had entirely ceased.

Experiment 6. An ounce and an half of saturated solution of muriate of barytes was injected into the stomach of a full grown cat, by means of an elastic gum tube. In a few minutes it operated as an emetic. The animal became giddy, afterwards insensible, and lay with dilated pupils, in general motionless, but with occasional convulsions. At the end of sixty-

five minutes, from the beginning of the experiment, he was apparently dead; but the heart was still felt through the ribs acting one hundred times in a minute. A tube was introduced into the trachea, and the lungs were inflated about thirty-six times in a minute; but the pulse sunk notwithstanding, and at the end of seven minutes the circulation had entirely ceased.

From these experiments I was led to conclude that the principal action of the muriate of barytes is on the brain; but in the first the pulse was feeble and intermitting; in the second, although the artificial respiration was made with the greatest care, the circulation could not be maintained more than a few minutes. These circumstances led me to suspect, that although this poison operates principally on the brain, it operates in some degree on the heart also. Further experiments confirmed this suspicion. In some of them the pulse soon became so feeble, that it could be scarcely felt; and its intermissions were more frequent; but in all cases the heart continued to act after respiration had ceased; and the cessation of the functions of the brain was therefore always the immediate cause of death. When I employed artificial respiration, after death had apparently taken place, I seldom was able to prolong the heart's action beyond a few minutes. In one case only it was maintained for three quarters of an hour. I never by these means succeeded in restoring the animal to life, although the experiments were made with the greatest care and in a warm temperature. In some instances, after the artificial respiration had been kept up for some time, there were signs of the functions of the brain being in some degree restored; but the pulse notwithstanding conti-

nued to diminish in strength and frequency, and ultimately ceased. I shall detail one of these experiments, as it serves to illustrate the double action of this poison on the nervous and vascular systems.

Experiment 7. Some muriate of barytes was applied to a wound in the side of a rabbit. The usual symptoms took place, and at the end of an hour the animal was apparently dead; but the heart still continued to contract. He was placed in a temperature of 80°, and a tube being introduced into the nostril, the lungs were artificially inflated about thirty-six times in a minute.

When the artificial respiration had been maintained for four minutes he appeared to be recovering; he breathed voluntarily one hundred times in a minute, and shewed signs of sensibility. The artificial respiration was discontinued. The voluntary respiration continued about nine minutes, when it had ceased, and the animal was again apparently dead; but the pulse continued strong and frequent. The lungs were again artificially inflated. At the end of four minutes the animal once more breathed voluntarily one hundred times in a minute, and repeatedly moved his limbs and eye-lids. The pulse became slower and more feeble.

In a few minutes the voluntary respiration again ceased, and the artificial respiration was resumed. The pulse had fallen to one hundred, and was feeble. The animal again breathed voluntarily; but he ceased to do so at the end of five minutes. The lungs were inflated as before; but he did not give any sign of life, nor was the pulse felt afterwards. On opening the thorax, the heart was found to have entirely ceased acting.

A probe having been introduced into the spinal marrow, it

was found that by means of the Voltaic battery powerful contractions might be excited, not only of the voluntary muscles, but also of the heart and intestines; from which it may be inferred, that the muriate of barytes, like arsenic, affects the circulation by rendering the heart insensible to the stimulus of the blood, and not by destroying altogether the power of muscular contraction.

The muriate of barytes affects the stomach, but in a less degree than arsenic. It operates as an emetic in animals that are capable of vomiting; but sooner when taken internally, than when applied to a wound. In general, but not constantly, there are marks of inflammation of the inner membrane of the stomach, but not of the intestine. In many instances there is a thin layer of dark coloured coagulum of blood lining the whole inner surface of the stomach and adhering very closely to it, so as to have a good deal of the appearance of a slough; and this is independent of vomiting, as where I met with it, it occurred in rabbits.

The same circumstances, from which it may be inferred that arsenic does not produce its deleterious effects until it has passed into the same circulation, leads to the same conclusion with regard to the muriate of barytes.

V. *On the Effects of the Emetic Tartar.*

The effects of the emetic tartar so much resemble those of arsenic and of muriate of barytes in essential circumstances, that it would be needless to enter into a detail of the individual experiments made with this poison.

When applied to a wound in animals, which are capable of vomiting, it usually, but not constantly, operates very speedily

as an emetic, otherwise I have found no material difference in the symptoms produced in the different species of animals, which I have been in the habit of employing as the subjects of experiment. The symptoms are paralysis, drowsiness, and at last complete insensibility; the pulse becomes feeble; the heart continues to act after apparent death; its action may be maintained by means of artificial respiration; but never for a longer period than a few minutes: so that it appears that this poison acts on the heart as well as on the brain; but that its principal action is on the latter. Both the voluntary and involuntary muscles may be made to contract after death, by means of Voltaic electricity. The stomach sometimes bears the marks of inflammation; but at other times it has its natural appearance. I have never seen any appearance of inflammation of the intestines. The length of time, which elapses from the application of the poison to the death of the animal, varies. In some instances it is not more than three quarters of an hour; but in others, it is two or three hours, or even longer.

When a solution of emetic tartar was injected into the stomach of a rabbit, the same symptoms took place as when it was applied to a wound.

VI. *On the Effects of the Corrosive Sublimate.*

When this poison is taken internally in very small and repeated doses, it is absorbed into the circulation, and produces on the system those peculiar effects, which are produced by other preparations of mercury. If it passes into the circulation in larger quantity, it excites inflammation of some part of the alimentary canal, the termination of which may vary accord-

ingly as it exists in a greater or less degree. When taken in a larger quantity still, it occasions death in a very short space of time. I had found, that if applied to a wounded surface, it produced a slough of the part, to which it was applied, without occasioning any affection of the general system. This led me to conclude that the effects of it, taken internally and in a large quantity, depended on its local action on the stomach, and were not connected with the absorption of it into the circulation. The following experiments appear to confirm this opinion.

Experiment 8. Six grains of corrosive sublimate, dissolved in six drams of distilled water, were injected into the stomach of a rabbit, by means of an elastic gum tube. No immediate symptoms followed the injection; the animal made no expression of pain; but in three minutes he became insensible; was convulsed; and in four minutes and an half, from the time of the injection being made, he died. Tremulous contractions of the voluntary muscles continued for some time afterwards. On opening the thorax, the heart was found to have entirely ceased acting, and the blood in the cavities of the left side was of a scarlet colour. The stomach was much distended. The pyloric and cardiac portions were separated from each other by a strong muscular contraction. The contents of the former were firm and solid, and in every respect resembled the usual contents of the stomach; while those of the cardiac portion consisted of the food of the animal much diluted by fluid; so that the solution, which had been injected, appeared to be confined to the cardiac portion of the stomach, and to be prevented entering the pyloric portion by the muscular contraction in the centre.

In the pyloric portion of the stomach the mucous membrane

had its natural appearance; but in the cardiac portion it was of a dark grey colour, was readily torn and peeled off; and in some parts its texture was completely destroyed; so that it appeared like a pulp, on removing which the muscular and peritoneal coats were exposed.

The repetition of the experiment was attended with similar results. The alteration of the texture of the internal membrane appears to have been occasioned by its being chemically acted on by the corrosive sublimate injected into it. When the injection is made into the stomach of a dead rabbit, precisely the same effects are produced, except that, as the middle contraction is here wanting, the appearances are not confined in the same degree to the cardiac portion.

Experiment 9. A scruple of corrosive sublimate, dissolved in six drams of distilled water, was injected into the stomach of a full grown cat. For the first five minutes no symptoms were produced. After this, the poison operated twice as an emetic. The animal appeared restless, and made expression of pain in the abdomen. He gradually became insensible, and lay on one side motionless, with the pupils of the eyes dilated. The respiration was laborious, and the pulse could not be felt. Twenty-five minutes after the poison was injected there was a convulsive action of the voluntary muscles, and death ensued. On opening the thorax immediately afterwards, the heart was seen still contracting, but very feebly.

The stomach was found perfectly empty and contracted. The mucous membrane was every where of a dark grey colour. It had lost its natural texture, and was readily torn and separated from the muscular coat. The internal membrane of the duodenum had a similar appearance, but in a less

degree, for nearly three inches from the pylorus. In the situation of the pylorus, the effects of the poison were less apparent than in any other part.

The particular state of the internal membrane of the stomach, in this experiment as well as in the last, appears to have been occasioned by the chemical action of the poison on it. When I injected a solution of corrosive sublimate into the stomach of a dead cat, and retained it there for a few minutes, a similar alteration of the texture of the internal membrane took place; but it assumed a lighter gray colour. The difference of colour may be explained by the vessels in the one case being empty, and in the other case being distended with blood at the time of the injection being made.

The destruction of the substance of the internal membrane of the stomach, precludes the idea of the poison having been absorbed into the circulation. We must conclude that death was the consequence of the chemical action of the poison on the stomach. This organ, however, is not directly necessary to life, since its functions, under certain circumstances, are suspended for hours, or even for days, without death being produced. Although the stomach was the part primarily affected, the immediate cause of death must be looked for in the cessation of the functions of one or more of those organs, whose constant action is necessary to life. From the scarlet colour of the blood in the left side of the heart, in the experiment on the rabbit, we may conclude that the functions of the lungs were not affected; but the affection of the heart and brain is proved by the convulsions, the insensibility, the affection of the pulse in both experiments, and the sudden cessation of the heart's action in the first, and we may therefore be

justified in concluding, that the immediate cause of death was in both of these organs. As the effects produced appear to have been independent of absorption, we may presume that the heart, as well as the brain, was acted on through the medium of the nerves.

That a sudden and violent injury of the stomach should be capable of thus speedily proving fatal is not surprising, when we consider the powerful sympathy between it, and the organs, on which life more immediately depends, and the existence of which many circumstances in disease daily demonstrate to us.

VII.

The facts which have been stated, appear to lead to the following conclusions respecting the action of the mineral poisons, which were employed in the foregoing experiments.

1. Arsenic, the emetic tartar, and the muriate of barytes do not produce their deleterious effects until they have passed into the circulation.

2. All of these poisons occasion disorder of the functions of the heart, brain, and alimentary canal; but they do not all affect these organs in the same relative degree.

3. Arsenic operates on the alimentary canal in a greater degree than either the emetic tartar, or the muriate of barytes. The heart is affected more by arsenic than by the emetic tartar, and more by this last than by the muriate of barytes.

4. The corrosive sublimate, when taken internally in large quantity, occasions death by acting chemically on the mucous membrane of the stomach, so as to destroy its texture; the organs more immediately necessary to life being affected in consequence of their sympathy with the stomach.

In making the comparison between them, we observe that the effects of mineral are less simple than those of the generality of vegetable poisons; and when once an animal is affected by the former, there is much less chance of his recovery than when he is affected by the latter.

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METEOROLOGICAL JOURNAL,

KEPT AT THE APARTMENTS

OF THE

ROYAL SOCIETY,

BY ORDER OF THE

PRESIDENT AND COUNCIL.

METEOROLOGICAL JOURNAL
for January, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Winds. Str.	Weather.
Jan. 1	9	30	45	30,26	68	NE	1	Cloudy.
	4,15	29	46	30,15	64	NE	1	Hazy.
2	9	28	44	30,03	63	NE	1	Cloudy.
	4	30	44	30,01	60	NNE	1	Cloudy.
3	10	26	42	29,69	68	E	1	Snow.
	4	24	43	29,74	64	E	1	Cloudy.
4	8,45	28	40	29,87	65	E	1	Cloudy.
	4,15	30	42	29,97	64	E	1	Cloudy.
5	9	25	38	29,84	63	E	2	Fair. Blowed hard in the night.
	4	28	42	29,80	58	E	1	
6	9,30	27	38	29,78	60	N	1	Cloudy.
	5	28	38	29,80	60	NE	1,2	Cloudy.
7	9	27	36	29,86	62	E	1	Fair.
	4	30	40	29,87	61	E	1	Fair.
8	9	27	36	29,81	63	E	1	Fair.
	4	28	40	29,77	62	E	1	Cloudy.
9	9	29	37	29,81	67	NNE	1	Cloudy. [morning.
	4	30	40	29,87	69	N	1	Cloudy. Some small snow in the
10	9	24	37	29,92	72		1	Foggy.
	6,30	35	41	29,85	72		1	
11	9,30	42	42	29,65	78	W	1,2	Rain.
	4,30	41	44	29,71	78	W	1	Fair.
12	9	41	42	29,61	77	S	2	Cloudy. Much rain in the night.
	4	45	45	29,44	78	SSW	2,3	
13	9,30	38	45	29,64	69	SW	1	Fair.
	5,45	42	45	29,64	74	S	1	Cloudy.
14	9	45	45	29,58	78	S	1	Cloudy.
	4	45	48	29,62	78	W	1	Cloudy.
15	9	44	48	29,58	75	W	1	Cloudy and rain.
	4	41	49	29,57	59	W	1	Fair.
16	9	35	47	29,85	67	W	1	Fair.
	4	40	49	29,98	63	W	1	Fair.

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1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Str.	Weather.
Jan. 17	9	45	48	29,74	76	SW	2	Rain.
	4	50	51	29,60	76	SW	1	Rain.
18	9	40	50	29,73	68	W	1	Cloudy.
	4,20	40	51	29,77	62	W	1	Cloudy.
19	9	45	48	30,31	66	WNW	1	Fair.
	4	42	51	30,39	65	NW	1	Fair.
20	9	35	48	30,38	71	S	1	Fair.
	5	37	48	30,25	67	SE	1	Fair.
21	9	35	45	30,04	70	S	1	Cloudy.
	4,30	39	47	30,05	71	WSW	1	Cloudy.
22	9	33	46	30,29	75	W	1	Cloudy and hazy.
	4,30	36	49	30,29	74	SW	1	Cloudy.
23	9	36	47	30,23	73	W	1	Hazy.
	4	40	50	30,28	74	W	1	Cloudy and very dark.
24	9	35	47	30,42	75	E	1	Fair.
	4,15	40	49	30,44	70	NNE	1	Fair.
25	9	36	48	30,51	71	NW	1	Hazy.
	4,40	36	49	30,48	65	NNE	1	Cloudy.
26	9	29	46	30,23	71	W	1	Fair, somewhat hazy.
	4,15	38	49	30,01	71	W	1	Cloudy.
27	9,30	40	48	29,68	70	W	1	Cloudy. Rain all night.
	9	34	46	29,66	64	W		Fair.
28	9	29	44	29,61	66	W	1	Fair, but somewhat hazy.
	5	33	48	29,61	60	W	1	Fair.
29	9	26	44	29,62	66	W	1	Foggy.
	4,20	33	48	29,67	60	W	1	Fair.
30	9	27	43	29,75	65	E	1	Foggy (very.)
	4,15	32	45	29,67	68	E	1	Cloudy.
31	9	34	43	29,14	77	E	1	Rain. Much snow in the night.
	4	43	47	29,09	78	S	1	Rain.

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1811	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Winds.		Weather.
	H. M.	o	o	Inches.	o	Points.	Str.	
Feb. 1	9	41	46	29,22	73	S	2	Fair. Blowed hard in the night.
	4	40	59	29,33	70	S	1	Cloudy.
2	8,45	41	47	29,61	75	E	1	Hazy. Rain in the night.
	4,30	42	50	29,47	73	E	1	Cloudy.
3	9,15	48	48	29,30	71	S	2	Fair. Blowed hard in the night.
	12	36	46	29,87	70		1	Fair.
4	9	36	46	30,03	71	S	1	Fair.
	4,30	46	51	30,07	69	S	1	Fair.
5	8,50	37	48	30,03	72	E	1	Fair.
	4,20	39	50	29,82	71	E	1	Cloudy.
6	9	47	49	29,68	72	S	1	Cloudy.
	4	48	52	29,52	69	SSW	1	Cloudy.
7	9	46	50	29,68	66	SW	1	Fair. Blowed hard in the night.
	4	45	54	29,67	65	SW	1,2	Fair.
8	9	45	51	29,69	68	S	2	Cloudy.
	4	48	53	29,48	73	SW	2	Rain.
9	9	43	52	29,59	72	E	1	Cloudy.
	4	46	54	29,70	66	W	1	Cloudy.
10	9	45	52	29,81	75	SW	1	Cloudy. Rain in the night.
	5	50	53	29,77	75	SW	1	Cloudy.
11	9,30	49	53	29,51	77	W	1	Cloudy. Rain in the night.
	4,25	49	55	29,32	70	WSW	1	Cloudy.
12	8,30	47	53	29,28	72	SW	1	Rain.
	4	47	56	29,08	71	W	1	Cloudy.
13	8,40	38	52	29,30	67	W	1	Cloudy.
	4	40	55	29,24	60	WNW	1	Fair.
14	9	38	52	29,21	66	NW	1	Cloudy.
	4	43	54	29,49	63	WNW	1	Cloudy.
15	8,40	38	52	29,68	60	NE	1	Hazy.
	4	40	53	29,44	62	E	1	Cloudy.
16	8,50	38	51	29,39	70	N	1,2	Cloudy.
	4	40	53	29,36	63	WSW	1	Cloudy.

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1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Str.	Weather.
Feb. 17	9	32	50	30,22	62	N	1	Hazy.
	4	37	49	30,18	62	E	1	Cloudy.
18	9	37	52	30,12	64	E	1	Cloudy.
	5	41	52	30,06	61	E	1	Rather hazy.
19	8,45	34	48	29,96	64	E	1	Fair.
	5,30	38	52	29,85	60	E	1	Fair.
20	8,40	35	48	29,67	67	E	1	Hazy.
	5,15	40	50	29,67	67	E	1	Hazy.
21	8,30	39	48	29,48	73	E	1	Hazy.
	5	44	51	29,00	70	E	1	Rain.
22	8,30	42	50	28,94	72	S	1	Fair.
	6,15	45	53	29,07	65	E	1	Fair.
23	8,25	41	51	29,18	72	S	1	Fair.
	6	44	52	29,19	70	S	1	Fine, with clouds.
24	8,30	43	51	29,11	74	SW	1	Rain. Much rain in the night.
	6	44	53	29,01	72	SW	1	Small rain.
25	8,30	39	51	29,19	71	W	1	Cloudy. Rain in the night.
	6	46	54	29,32	64	W	1	Fair.
26	8,30	48	52	29,18	75	SW	1	Cloudy. Rain in the night.
	6	49	54	29,27	64	W	1	Fair.
27	8,32	45	52	29,46	66	W	1	Fair.
	6,15	47	55	29,65	60	W	1	Fair.
28	9	47	53	29,73	71	S	1	Cloudy.
	6	50	55	29,49	73	SW	1	Rain.

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1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Winds. Str.	Weather.
Mar. 1	8,30	42	54	29,58	65	W	1	Fair.
	4,35	40	52	29,78	67	W	1	Fine.
2	8	46	53	29,79	73	W	1	Hazy.
	6	49	55	29,85	65	W	1	Fair.
3	9	49	54	29,76	73	W	1	Rain.
	4,25	44	53	30,00	68	W	1	Fair.
4	8,45	48	53	29,99	73	W	1	Cloudy.
	6,35	54	57	29,97	61	W	1	Fair.
5	8,30	46	54	29,76	71	S	2	Cloudy.
	6	50	55	29,40	66	S by W	1	Fair.
6	8,35	43	54	29,45	65	W	1	Fair.
	7,45	47	55	29,52	70	W	2	Rain.
7	8,45	50	55	29,34	69	SW	2	Cloudy.
	6	48	56	29,24	70	SW	1	Rain.
8	8,30	48	55	29,29	79	N	2	Rain.
	6,15	42	46	29,48	70	N	2	Rain.
9	8,45	39	53	30,21	65	N	1	Fair.
	7,45	44	56	30,40	57	NE	1	Fair.
10	9,45	45	53	30,46	64	SW	1	Cloudy.
	7,15	50	54	30,45	65	W	1	Cloudy.
11	8,30	44	53	30,45	70	W	1	Foggy.
	2	49	54	30,40	68	SW	1	Hazy.
12	8,20	47	54	30,43	73	E	1	Foggy.
	2,15	51	56	30,38	61	E	1	Cloudy.
13	8,30	42	53	30,34	70	NE	1	Cloudy.
	3,45	49	58	30,28	58	ESE	1	Fair.
14	8,20	43	53	30,32	68	E	1	Hazy.
	3,30	47	58	30,31	61	E	1	Fair.
15	8,30	40	54	30,33	67	E	1	Fair.
	4	45	60	30,34	61	E	1	Fair.
16	8,30	40	54	30,28	64	E	1	Fair.
	2	48	57	30,21	55	E	1	Fair.

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1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer.	Winds. Points.	Winds. Str.	Weather.
Mar. 17	8,45	41	54	30,13	° 69	E	1	Foggy.
	4,45	50	64	30,11	52	S	1	Fair.
18	8,20	41	53	30,11	65	SE	1	Foggy.
	3,30	52	59	30,09	60	S	1	Fair.
19	8,20	47	55	30,16	73	W	1	Foggy.
	3,45	53	58	30,17	60	W	1	Cloudy.
20	9	48	56	30,11	69	W	1	Cloudy.
	2,10	52	56	30,07	67	W	1	Cloudy.
21	8,30	50	56	29,99	70	SW	1	Cloudy. Rain in the night.
	3,50	57	58	29,97	65	W	1	Cloudy.
22	9	48	57	30,11	70	N	1	Rain.
	8,30	45	55	30,29	61	N	1	Fair.
23	8,20	40	55	30,37	67	NNE	1	Fair.
	3,30	49	59	30,45	58	W	1	Fair.
24	9	46	56	30,34	70	E	1	Fair.
	2,30	50	58	30,25	57	E	1	Fair.
25	8	44	50	30,16	71	E	1	Hazy.
	2,45	49	58	30,10	57	E	1	Fair.
26	9	46	56	30,06	70	E	1	Fair.
	3	50	59	30,08	61	E	1	Fair.
27	8,45	44	55	30,31	66	E	1	Fair.
	4	49	54	30,37	72	E	1	Fair.
28	9	40	54	30,46	64	ENE	1	Foggy.
	3	51	57	30,46	50	NE	1	Fair.
29	8,45	47	56	30,56	66	N	1	Fair.
	2	53	58	30,51	56	E	1	Fair.
30	8	40	55	30,34	68	E	1	Foggy.
	2	52	58	30,23	54	W	1	Fair.
31	9	47	56	30,16	69	E	1	Cloudy.
	6	47	55	30,12	64	W	1	Cloudy.

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1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Winds. Str.	Weather.
April 1	8	45	55	30,65	65	E	1	Hazy.
	2,45	50	58	29,98	62	E	1	Fair.
2	8	45	55	29,83	71	E	1	Hazy.
	2	59	58	29,83	57	S	1	Cloudy.
3	8	50	57	29,93	70	W	1	Fair.
	3,45	58	60	29,93	55	NNW	1	Cloudy.
4	8	51	58	30,00	68	N	1	Fair.
	2,30	57	60	30,05	60	W	1	Cloudy.
5	8	44	58	30,10	68	E	1	Cloudy.
	3,30	49	59	30,04	62	W	1	Fair.
6	8	37	57	29,91	67	E	1	Cloudy.
	3	51	59	29,71	60	SE	1	Fair.
7	9,30	41	56	29,48	65	S	1	Rain.
	2	43	56	29,48	56	N	1	Cloudy.
8	8,40	37	53	29,53	56	NE	1	Cloudy.
	4,30	42	54	29,54	56	N	1	Cloudy.
9	8,30	38	51	29,60	55	N	1	Fair.
	4	44	57	29,64	48	NE	1	Fair.
10	8,40	39	51	29,80	56	NNW	1	Fair.
	3	45	54	29,83	50	SSW	1	Fair.
11	8,45	40	51	29,98	67	N	1	Rain.
	2,30	47	54	30,05	54	NE	1	Fair.
12	8,30	44	52	30,17	63	SSW	1	Fair.
	2	48	53	30,18	52	SW	2	Cloudy.
13	9	50	52	29,93	76	WSW	1	Rain.
	2,30	60	56	29,98	62	W	1	Cloudy.
14	9	54	55	30,12	69	W	1	Cloudy.
	2	62	56	30,14	60	W	1	Cloudy.
15	9	55	56	30,15	72	W	1	Cloudy.
	3	59	56	30,12	60	W	1	Cloudy.
16	8,45	53	56	29,92	72	SSW	1	Cloudy.
	2,30	57	56	29,83	68	W	1	Rain.

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for April, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Str.	Weather.
Apr. 17	8,30	47	55	29,28	69	NE	1	Cloudy.
	3,40	56	58	29,63	56	S	1	Fair.
18	9,10	50	56	29,17	67	W	1	Rain.
	2,15	56	58	29,17	59	SW	1	Cloudy.
19	8,45	50	56	29,19	66	W	1	Fine.
	2	54	58	29,12	64	SW	1	Fair.
20	8,45	52	56	29,28	65	SSW	2	Cloudy.
	3,45	57	59	29,33	61	SW	1	Cloudy.
21	8	52	57	30,39	68	SSW	1	Cloudy. Blowed hard in the night.
	5,20	57	59	29,54	55	S	1	Fair.
22	8,30	55	58	29,61	68	W	1	Cloudy.
	3	62	62	29,55	62	E	1	Fair.
23	8,30	60	60	29,56	66	S	1	Cloudy.
	2,30	67	63	29,59	53	E	1	Fair.
24	8,30	57	61	29,73	70	W	1	Hazy.
	3,15	65	65	29,75	60	E	1	Fair.
25	8,30	57	61	29,77	62	E	1	Cloudy.
	2,30	61	62	29,74	58	E	1	Fair.
26	8,45	55	61	29,73	67	N	1	Rain.
	3	61	63	29,67	54	E	1	Fair.
27	8,45	56	60	29,55	60	NNE	1	Fine.
	3,30	59	63	29,55	50	E	1	Fair.
28	9,15	54	60	29,58	59	SW	1	Fair.
	2	60	61	29,58	55	SSW	1	Fair.
29	8,30	51	59	29,32	61	SSW	1	Rain.
	3,30	52	60	29,39	63	W	1	Rain.
30	8,30	52	59	29,77	62	W	1	Cloudy.
	2,30	56	59	29,78	58	WSW	1	Cloudy.

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for May, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Str.	Weather.
May 1	8,15	55	58	29,65	74	SW	1	Rain.
	3,20	60	59	29,62	70	WSW	1	Rain.
2	8,30	55	59	29,60	68	W	1	Cloudy.
	3	59	60	29,73	64	W	1	Cloudy.
3	8,50	53	58	30,01	67	W	1	Cloudy.
	2,45	58	59	29,98	61	SW	1	Cloudy.
4	8,45	55	58	29,87	72	SW	1	Cloudy.
	4	63	61	29,90	63	W	1	Cloudy.
5	8,45	55	59	29,68	72	WSW	1	Cloudy.
	4	54	57	29,85	66	SW	1	Cloudy.
6	8,20	48	58	30,04	60	E	1	Cloudy.
	3,30	51	58	30,00	61	E	1	Cloudy.
7	8,30	51	57	29,70	70	W	1	Fair.
	6,15	52	58	29,70	69	WSW	1	Rain.
8	8,30	49	57	29,80	69	E	1	Cloudy.
	3,15	57	58	29,73	59	S	1	Cloudy.
9	8,30	52	57	29,52	72	ESE	1	Rain.
	3,45	55	57	29,48	70	W	1	Cloudy.
10	9	54	57	29,72	65	S	2	Cloudy.
	3	61	58	29,72	59	S	1	Cloudy.
11	8,45	55	57	29,81	66	S	1	Cloudy.
	3	63	59	29,78	59	SSW	1	Cloudy.
12	8,30	62	59	29,68	67	W	1	Fair.
	2,30	68	62	29,61	57	SSW	1	Fair.
13	8,45	64	62	29,51	64	W	1	Fine.
	4,30	69	67	29,48	56	S	1	Fair.
14	8,15	64	64	29,44	61	W	1	Cloudy.
	2,45	63	64	29,55	58	SSW	1	Cloudy.
15	8,25	57	61	29,75	62	E	1	Fair.
	2,30	64	64	29,73	56	W	1	Fair.
16	8,30	60	62	29,76	66	E	1	Cloudy.
	2,45	64	64	29,78	59	SSW	1	Cloudy.

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for May, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Winds. Str.	Weather.
May 17	8,20	56	62	29,87	59	SE	1	Fine.
	3,40	64	65	29,84	51	NE	1	Fair.
18	8,45	60	62	29,84	67	E	1	Rain.
	2,40	66	64	29,83	59	SE	1	Cloudy.
19	8, 5	61	64	29,92	74	NNE	1	Rain.
	3	58	63	29,94	70	N	1	Cloudy.
20	8,30	57	61	29,88	67	N	1,2	Cloudy.
	3	67	66	29,78	57	E	1	Fair. [and rain the night.
21	8,30	57	62	29,67	75	E	1	Rain. Much thunder, lightning,
	3,15	62	63	29,68	69	N	1	Cloudy.
22	8,30	61	63	29,68	73	E	1	Cloudy.
	3	66	65	29,63	63	E	1	Cloudy.
23	8,45	59	62	29,80	59	W	1	Cloudy.
	2,50	65	64	29,86	51	S by W	1	Fine.
24	9	60	62	29,88	61	SE	1	Rain.
	3	66	64	29,84	59	SSW	1	Fine.
25	8,30	59	63	29,96	64	S	1	Cloudy.
	3,15	66	65	30,00	56	SSW	1	Fine.
26	9,15	66	64	30,01	62	E	1	Fine.
	4	72	69	29,98	54	SW	1	Cloudy.
27	8,40	65	66	29,94	67	E	1	Hazy.
	3,15	71	69	29,83	56	E	1	Fine.
28	8,45	62	66	29,61	67	SW	2	Cloudy.
	3	65	67	29,70	55	SW	1,2	Fine.
29	8,30	57	64	29,68	61	W	2	Cloudy.
	3,15	59	64	29,77	63	W	2	Rain.
30	8,45	57	63	29,95	62	W	1	Fine.
	3.30	67	66	29,90	53	SSE	1	Fine.
31	8,30	58	63	29,47	69	NNE	1	Rain. Thunder and lightning.
	3,10	65	65	29,41	65	S by E	1	Rain.

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for June, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within °	Barom. Inches.	Hygrometer. °	Winds. Points.	Winds. Str.	Weather.
June 1	9	60	64	29,69	61	S	1,2	Cloudy.
	4	65	64	29,70	57	S by E	1	Cloudy.
2	9	58	63	29,48	72	E	1	Rain.
	3,30	60	63	29,41	57	S	2	Cloudy.
3	8,25	55	62	29,65	66	W	1	Cloudy.
	3	63	63	29,83	52	W	1	Fine.
4	9	58	62	29,93	62	SW	2	Cloudy.
	4	63	63	29,88	58	S by W	2	Cloudy.
5	8,30	57	61	29,63	70	S	2	Rain. Blowed hard all night.
	3	57	62	29.63	65	W by N	1	Rain.
6	8,20	58	61	29,78	62	SW	1	Cloudy.
	3,15	64	63	29,80	56	SW	1	Fine.
7	9	60	61	29,93	65	SW	1	Cloudy.
	3	66	63	29,9[illegible]	60	SSW	1	Cloudy.
8	8,30	67	62	29,86	62	E	1	Fair. [at 6.
	3,15	71	69	29,75	60	S	1	Fine and clear. A thunder-storm
9	8,45	59	63	29,11	59	SW	1	Cloudy.
	2,20	66	64	30.15	51	W	1	Fine.
10	8,30	58	62	30,10	60	E	1	Cloudy.
	3,30	67	64	29,97	51	S	1	Fine.
11	8,40	62	60	29,91	56	W	1	Fine.
	3,45	65	64	29.92	54	SW	1	Cloudy.
12	8,10	58	60	29,91	61	SW	1	Fine.
	3	65	64	29,89	52	SW	1	Fine.
13	8,30	55	61	30.10	59	W	1	Fine.
	3	63	63	30,11	53	W	1	Fine.
14	8,30	57	61	30,04	59	S	1	Cloudy.
	3,30	64	63	29,92	52	S	1	Fine.
15	8,15	59	61	29,83	58	W	1	Fine.
	3,30	67	65	29,90	53	W	1	Fine.
16	9	61	63	29,97	60	W	1	Cloudy.
	4	65	65	29,96	55	W	1	Cloudy, with snow.

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1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hy-gro-me-ter.	Winds. Points.	Str.	Weather.
					°			
June 17	8,30	60	62	30,17	53	S	1	Cloudy.
	3,30	65	65	30,25	52	N	1	Cloudy.
18	8,45	62	63	30,38	51	E	1	Fine.
	3	68	66	30,31	52	E	1	Fine.
19	8,40	62	64	30,21	59	S	1	Fair.
	3,30	70	68	30,09	51	W by N		Fine.
20	8,45	63	65	29,90	56	S	1	Cloudy.
	3	58	64	29,80	56	N	1	Cloudy.
21	8,45	57	63	29 72	58	N	1	Cloudy.
	3,15	59	63	29,74	55	NE	1	Cloudy.
22	9,30	58	61	29,87	58	N	1	Fine.
	3,30	59	62	29,88	55	N	1	Cloudy.
23	8,30	55	60	29,87	68	NE	1	Rain.
	3,10	59	61	29,84	59	N	1	Cloudy.
24	9,30	57	61	29,70	72	N	1	Cloudy.
	3,30	64	63	29,67	62	E	1	Fine.
25	8,30	62	62	29,78	60	E	1	Fine.
	3,30	67	66	29,82	54	E	1	Fine.
26	9	67	64	29 91	60	E	1	Fine.
	3	71	67	29,91	52	N	1	Cloudy.
27	9	61	64	29,83	74	S	1	Rain.
	3	66	65	29,87	65	W	1	Cloudy.
28	8,45	64	65	29,91	75	N	1	Cloudy.
	3,30	66	66	29,89	72	N	1	Cloudy.
29	9	63	65	29,84	76	N	1	Cloudy.
	4	67	66	29,83	68	N	1	Cloudy.
30	9,30	63	65	29,83	62	NE	1	Cloudy.
	4	66	66	29,83	70	N	1	Cloudy.

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for July, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Str.	Weather.
July 1	8,20	62	65	29,80	74	N	1	Cloudy.
	3	65	65	29,80	67	N	1	Cloudy.
2	8,30	64	65	29,84	68	N	1	Cloudy.
	3,40	69	67	29,85	62	E	1	Cloudy.
3	8,15	62	65	29,94	72	NNE	1	Cloudy.
	4	63	65	30,02	70	NNE	1	Cloudy.
4	8,30	67	63	30,09	64	N	1	Cloudy.
	3,15	64	63	30,09	62	N	1	Cloudy.
5	8,20	57	62	30,13	60	NW	1	Hazy.
	3,10	64	63	30,10	54	NW	1	Cloudy.
6	8,15	58	62	30,04	64	N	1	Hazy.
	3,15	65	65	30,00	55	W	1	Cloudy.
7	8,50	62	63	29,95	65	NW	1	Cloudy.
	3,30	63	63	29,91	62	N	1	Cloudy.
8	8,25	58	62	29,91	62	N	1	Cloudy.
	3,30	67	65	29,90	55	N	1	Cloudy.
9	8,30	62	63	29,99	65	E	1	Cloudy.
	3	68	65	30,00	58	E	1	Cloudy.
10	8,30	62	64	30,06	66	E	1	Cloudy.
	3,30	69	67	30,04	57	W	1	Cloudy.
11	8,20	65	65	30,11	66	NW	1	Cloudy.
	2,20	72	69	30,10	55	W	1	Cloudy.
12	8	63	67	30,10	65	W	1	Fine.
	4,15	72	71	30,04	55	W	1	Cloudy.
13	8,20	64	68	29,97	67	W	1	Cloudy.
	3,20	72	70	29,84	58	W	1	Cloudy.
14	9	64	68	29,86	68	SW	1	Cloudy.
	3,30	65	67	29,83	63	SW	1	Cloudy.
15	8.45	64	67	29,91	62	SW	1	Cloudy.
	3,30	70	68	29,89	62	SSW	1	Cloudy.
16	7,45	64	67	29,86	67	SW	1	Cloudy.
	3,30	69	68	29,86	55	W	1	Cloudy.

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1811	Time.	Therm. without.	Therm. within.	Barom.	Hy-gro-me-ter.	Winds.		Weather
	H. M	°	°	Inches.		Points.	Str.	
					°			
July 17	8,30	63	61	29.90	65	SW	1	Cloudy.
	3	69	68	29,88	57	SW	1	Cloudy.
18	8,30	64	66	29,79	67	S	1	Cloudy.
	3	70	68	29,80	58	S	1	Cloudy. [in the night.
19	8	67	67	29,67	72	S	1	Cloudy. Thunder and lightning
	4	65	67	29,84	63	SW	1	Cloudy.
20	8	63	67	29,95	65	N by W	1	Cloudy.
	4	63	67	29,96	64	W	1	Cloudy,
21	8	56	65	29,99	74	W	1	Rain.
	3,50	59	63	29,90	73	NW	1	Rain.
22	8	60	64	29,94	75	W	1	Rain.
	3,15	68	67	29.98	60	W	1	Cloudy.
23	8,30	60	64	30,08	65	W	1	Cloudy.
	3,20	68	66	30,07	55	W	1	Cloudy.
24	8,40	62	65	30,18	57	N	1	Hazy.
	3,15	68	68	30,18	54	SW	1	Fair.
25	8,30	62	65	30,23	65	W	1	Fine, some clouds.
	3,40	70	68	30,23	59	NW	1	Cloudy.
26	8,30	62	66	30,23	68	W	1	Cloudy.
	3,15	61	68	30,21	70	W	1	Cloudy.
27	8,15	63	67	30,17	63	N	1	Hazy.
	3,25	71	69	30,09	57	E	1	Cloudy.
28	8,15	65	67	30,02	62	E	1	Cloudy.
	3.45	73	72	29,96	55	SE	1	Fair.
29	8,45	68	69	29.96	54	N	1	Fine.
	3,15	73	71	29,97	57	N	1	Fine, with some clouds.
30	8,30	64	69	30,15	61	N	1	Hazy.
	3,30	68	69	30,14	53	N	1	Cloudy.
31	8	62	67	30,17	60	N	1	Fine.
	4,15	65	67	30,07	56	N	1	Cloudy.

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for August, 1811.

1811	Time. H. M.	Therm. without. o	Therm. within. o	Barom. Inches.	Hygrometer.	Winds. Points.	Winds. Str.	Weather.
Aug. 1	8,20	64	64	30,02	o 62	N	1	Cloudy.
	4	68	68	30,60	56	W	1	Cloudy.
2	8,15	62	67	29,97	63	SW	1	Cloudy.
	3,30	69	69	29,87	55	S	1	Fair.
3	8,45	66	67	29,77	61	S	1	Cloudy.
	3,30	71	69	29,66	56	S	1	Cloudy.
4	9	63	66	29,72	60	W	1	Hazy.
	2,45	65	67	29,77	54	NW	1	Fine.
5	8,40	59	65	29,64	69	S	1	Cloudy.
	3,30	66	67	29,68	58	W	1	Cloudy.
6	8,30	60	65	29,68	64	SW	1	Rain.
	3,30	66	67	29,68	64	SW	1	Rain.
7	8	58	64	29,68	64	SW	1	Rain.
	3,30	66	67	29,70	54	W	1	Fair.
8	8,30	58	63	29,51	68	S	1	Rain.
	3,10	61	65	29,46	62	NW	1	Fair.
9	8,30	58	62	29,56	64	W	1	Cloudy.
	3,45	60	64	29,56	64	NW	1	Rain.
10	8,45	58	62	29,66	61	N	1	Cloudy.
	3,20	60	62	29,71	53	NW	1	Cloudy.
11	8,30	55	60	29 94	59	N	1	Fine.
	3,15	63	61	30,01	54	N	1	Cloudy.
12	8,30	55	60	30,29	61	W	1	Cloudy.
	5,15	63	62	30,21	56	W	1	Cloudy.
13	8,30	59	61	30,19	70	S	1	Rain.
	4,30	67	63	30,13	63	W	1	Cloudy.
14	8,20	60	62	30,29	65	N	1	Cloudy.
	3,45	66	65	30,29	56	NW	1	Cloudy.
15	8,45	58	61	30,33	63	W	1	Cloudy.
	3 20	65	65	30,24	55	W	1	Fair.
16	8,45	60	61	30,00	64	W	1	Cloudy.
	2,20	65	64	30,02	62	W	1	Fine.

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1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Str.	Weather.
Aug. 17	9	59	61	30,12	62	W	1	Fine.
	4	66	66	30,13	55	NW	1	Fine.
18	9	62	63	30,19	62	E	1	Fair.
	3,15	68	67	30,13	58	E	1	Fine.
19	8,45	62	64	29,81	67	E	1	Rain.
	3,30	66	65	29,70	65	SW	1	Cloudy. Much thunder and lightning in the forenoon.
20	8,45	57	64	29,80	67	W	1	Rain.
	3,30	64	65	29,91	53	W	1	Cloudy.
21	8,45	57	59	30,09	60	W	1	Fine.
	3	66	64	30,13	56	W	1	Cloudy.
22	8,45	62	63	30,13	67	W	1	Cloudy.
	3,45	67	66	30,08	62	W	1	Cloudy.
23	8,45	60	64	30,00	70	W	1	Hazy.
	3,45	66	65	29,91	62	SW	1	Cloudy.
24	8,45	60	63	29,77	68	S	1	Cloudy.
	4,15	67	67	29,68	60	S by E	1	Cloudy.
25	9,30	62	64	29,57	70	W	1	Rain.
	3,30	65	65	29,55	65	S	1	Cloudy.
26	8,45	58	62	29,75	63	W	1	Fair.
	3,15	65	66	29,79	55	SW	1	Fine, some clouds.
27	8,45	63	64	29,79	62	SW	2	Fair.
	3,15	65	64	29,81	66	SW	2	Cloudy.
28	8,45	57	62	30,15	64	W	1	Fine.
	3	64	64	30,14	55	SW	1	Cloudy.
29	8,30	59	62	30,07	63	S	1	Fair.
	4,45	64	64	30,02	62	SW	1,2	Rain.
30	8,45	58	62	30,22	63	W	1	Fine.
	4	65	67	30,24	55	W	1	Fine.
31	9	59	62	30,18	63	SSW	1	Fine.
	3,30	66	68	30,09	57	W	1	Fine.

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1811	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Winds.		Weather.
	H. M.	°	°	Inches.	°	Points.	Str.	
Sep. 1	9,30	60	64	30,13	59	NW	1	Cloudy.
	4,25	64	67	30,15	55	NW	1	Fine.
2	9	60	63	30,20	54	N	1	Cloudy.
	3,30	62	65	30,20	55	N	1	Cloudy.
3	8,45	59	62	30,35	60	E	1	Fair.
	3,45	62	63	30,30	54	E	1	Cloudy.
4	8,50	58	61	30,31	60	N	1	Cloudy.
	3,15	62	62	30,26	56	W	1	Cloudy.
5	8,45	60	62	30,25	66	N	1	Cloudy.
	3,40	65	68	30,26	55	E	1	Fine.
6	8,30	62	63	30,25	68	E	1	Fair.
	3	66	68	30,22	52	E	1	Cloudy.
7	8	61	64	30,26	67	E	1	Hazy.
	3,30	67	70	30,25	48	E	1	Fair.
8	9	58	64	30,27	70	N	1	Cloudy.
	3,30	64	69	30,25	55	E	1	Fair.
9	8,30	60	65	30,30	70	N	1	Hazy.
	3,55	68	73	30,26	53	E	1	Fair.
10	8,45	62	65	30,24	70	NNE	1	Hazy.
	3	67	70	30,20	57	E	1	Fine.
11	8,45	64	66	30,16	66	N	1	Fine.
	3	72	72	30,12	52	W	1	Fair.
12	8,30	66	68	30,23	65	W	1	Hazy.
	3,30	72	72	30,26	66	NE		Fine.
13	8,45	63	67	30,29	67	E	1	Fine.
	3	66	70	30,21	59	E	1	Fair.
14	8,25	58	65	30,11	65	N	1	Hazy.
	4,30	67	80	30,09	55	E	1	Fair.
15	8,30	64	67	30,13	69	E	1	Cloudy.
	3,10	65	70	30,13	60	E	1	Fair.
16	8,20	61	66	30,19	65	E	1	Cloudy.
	6	62	72	30,13	60	E	1	Fair.

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for September, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Str.	Weather.
Sep. 17	8,45	60	65	30,07	69	E	1	Hazy.
	2	64	69	30,01	66	E	1	Fair.
18	8,45	62	65	30,00	65	E	1	Fair.
	3,10	64	68	29,98	51	E	1	Fair.
19	8,20	60	64	29,91	68	N	1	Cloudy.
	3	65	68	29,84	58	E	1	Cloudy.
20	8,45	62	65	29,64	69	E	1	Hazy.
	3,35	67	68	29,54	60	S	1	Cloudy.
21	8,30	62	65	29,60	68	S	1	Foggy.
	3,45	64	65	29,63	63	SE	1	Rain.
22	8,40	61	64	29,89	67	W	1	Cloudy.
	3,20	63	67	29,89	61	S	1	Fair.
23	8,35	61	60	29,64	70	S	2	Rain. Much wind in the night.
	3,25	61	65	29,53	63	W	2	Cloudy.
24	8,40	54	62	29,70	66	W	1	Fair.
	3,30	60	65	29,66	58	S	1	Fine.
25	8,30	57	62	29,98	72	SW	2	Rain.
	3,20	56	62	29,93	70	W	2	Rain.
26	8,30	50	59	29,84	63	W	1	Cloudy.
	3	58	62	29,81	62	SW	1	Cloudy.
27	8,20	47	58	29,78	67	W	1	Cloudy.
	4	50	58	29,26	71	W	1	Rain.
28	8,20	45	55	29,44	72	W	1	Cloudy.
	4,30	56	58	29,43	67	N	1	Cloudy.
29	9	53	57	29,63	68	W	1	Cloudy.
	3,15	59	59	29,66	61	N	1	Fair.
30	8,10	52	57	29,79	70	W	1	Fine.
	3	59	59	29,81	64	S	1	Fine.

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for October, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer.	Winds. Points.	Str.	Weather.
Oct. 1	8,45	58	59	29,60	° 72	SSW	2	Rain.
	3,20	58	61	29,55	65	W	2	Rain.
2	8,30	54	57	29,84	69	W	1	Fair.
	3,20	59	61	29,81	61	W	1	Fine.
3	8,30	50	57	29,92	69	N	1	Foggy.
	4,30	57	58	29,70	71	W	1	Rain.
4	8,30	60	60	29,50	70	E	1	Rain.
	4,30	62	65	29,52	63	SE	1	Cloudy.
5	8,20	61	61	29,65	70	S	2	Rain.
	3	62	63	29,67	63	S	2	Fine.
6	9	57	60	29,88	67	W	1	Cloudy.
	3,15	60	62	29,96	57	W	1	Fair.
7	8,30	61	61	29,94	77	W	1	Cloudy.
	3,30	64	62	29,96	69	W	1	Cloudy.
8	8,20	58	61	30,00	74	W	1	Fine.
	1	61	62	29,99	63	W	1	Fine.
9	9	59	63	30,06	71	WSW	1	Cloudy.
	4	61	62	30,04	68	SW	1	Cloudy.
10	8,30	57	61	30,09	74	SW	1	Cloudy.
	4,30	63	62	30,06	66	W	1	Cloudy.
11	8,30	59	62	29,93	71	S	1	Cloudy.
	3,30	63	65	29,85	65	SW	1	Cloudy.
12	8,30	60	62	29,67	71	SW	1	Cloudy.
	4	63	65	29,74	68	SW	1	Cloudy.
13	8,30	57	61	29,82	74	SW	1	Rain.
	4	60	62	29,83	66	SW	1	Cloudy.
14	8,30	58	61	29,83	69	SW	1	Cloudy.
	4	60	62	29,85	68	SW	1	Cloudy.
15	8,40	58	61	29,88	74	SW	1	Cloudy.
	4	65	67	29,85	63	S	1	Fair.
16	7,45	60	63	29,85	73	SW	1	Fine.
	3	64	65	29,92	65	W	1	Cloudy.

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for October, 1811.

1811	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Winds.		Weather.
	H. M.	°	°	Inches.	°	Points.	Str.	
Oct. 17	8,30	60	63	30,12	74	S	1	Cloudy.
	3	65	63	30,14	63	S	1	Cloudy.
18	8	56	62	30,19	74	W	1	Foggy.
	4	64	65	30,19	68	SW	1	Cloudy.
19	8,20	60	63	30,22	73	W	1	Rain.
	2,32	63	64	30,24	68	E	1	Cloudy.
20	8,30	57	62	30,28	73	E	1	Foggy.
	3	62	64	29,19	66	S	1	Fine.
21	9	57	62	30,00	67	S	1	Cloudy.
	4	62	62	29,82	70	S	1	Cloudy.
22	8,30	60	62	29,58	71	S	1	Cloudy.
	4,30	62	63	29,53	68	SE	1	Rain.
23	9	55	60	29,58	68	SW	1	Fine.
	4	60	62	29,57	67	S	1	Hazy.
24	8,30	53	59	29,53	67	S		Cloudy.
	3	57	59	29,50	66	SW	1	Cloudy.
25	8,30	53	58	29,49	68	W	1	Hazy.
	4	56	58	29,34	70	SE	1	Cloudy.
26	8,20	48	56	28,71	70	SE	1	Cloudy.
	3	50	57	28,73	65	S	2	Rain.
27	8,30	48	54	28,88	71	E	1	Cloudy.
	4	51	57	28,86	66	SE	1	Cloudy.
28	8,20	49	55	28,84	70	S	1	Hazy.
	3,30	51	56	28,88	69	S	1	Rain. [and rain in the night.
29	8,10	45	52	28,83	69	SW	1	Cloudy. A violent squall of wind
	3,20	52	54	28,96	63	W	1	Cloudy.
30	8,30	48	53	29,06	72	E	1	Rain.
	4,45	53	57	29,05	70	W	1	Rain.
31	8,45	48	54	29,64	71	W	1	Fair.
	3,30	55	57	29,79	65	W	1	Cloudy.

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for November, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hy-gro-me-ter. °	Winds. Points.	Str.	Weather.
Nov. 1	8,40	57	57	29,77	74	S by W	2	Rain.
	3,20	60	59	29,75	76	SW	1	Cloudy.
2	9	61	60	29,68	73	S	2	Rain.
	4	60	60	29,66	72	S	1	Cloudy.
3	9	55	59	29,67	69	W	1	Fine. Blowed hard in the night.
	3,20	57	59	29,79	64	SW	1	Cloudy.
4	8,30	51	58	29,72	73	W	1	Fine.
	10	48	58	30,04	67	W	1	Fine.
5	8,30	52	57	29,89	70	S by W	2	Rain.
	3,30	53	59	29,91	71	W	1,2	Fine.
6	8,30	48	57	29,92	68	W	1	Cloudy.
	4	53	58	29,78	69	W	1	Cloudy.
7	9	49	56	29,55	71	W by N	1	
	4	50	58	29,61	62	W	1	Fair.
8	9	45	57	29,48	68	E	1	Rain.
	3,30	49	57	29,88	74	E	1	Rain.
9	9	48	56	29,67	75	W	1	Foggy.
	4	52	59	29,64	72	SSW		
10	9	51	57	29,36	75	E	1	Rain.
	3,45	50	58	29,84	67	S	1	Fair.
11	8,30	43	55	29,79	73	W	1	Fair.
	4	51	58	29,36	69	W	1	Fair.
12	9	42	55	29,97	65	W	1	Fair.
	4	48	57	29,89	62	W	1	Cloudy.
13	9	50	56	29,57	75	W	1	Cloudy.
	3	50	58	29,64	64	W	1	Rain.
14	9	45	54	29,74	68	S	1	Cloudy.
	4	52	58	29,72	67	S	1	Cloudy.
15	8,45	42	55	29,69	66	W	1	Cloudy.
	5,40	43	57	29,58	65	NW	1	Fair.
16	9	43	53	29,55	67	N	1	Hazy.
	4	48	55	29,65	65	W	1	Cloudy.

METEOROLOGICAL JOURNAL

for November, 1811.

1811	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Winds.		Weather
	H. M.	°	°	Inches.	°	Points.	Str.	
Nov. 17	9,30	44	53	30,11	64	NW	1	Cloudy.
	3	48	52	30,18	63	WNW	1	Cloudy.
18	9	48	52	30,28	73	W	1	Foggy.
	4	50	55	30,31	73	W	1	Rain.
19	8,45	49	53	30,26	73	W	1	Cloudy.
	4	49	56	30,32	63	N	1	Fine.
20	8,40	40	53	30,40	68	N	1	Hazy.
	3,30	44	55	30,40	66	N	1	Fine.
21	8,45	34	52	30,35	68	W	1	Foggy.
	4	42		30,28	62	N	1	Cloudy.
22	9	40	53	30,27	69	E		Foggy.
	3,45	44	54	30,23	62	NE	1	Cloudy.
23	8,30	32	51	30,23	68	N	1	Foggy.
	4	41	52	30,23	70	N	1	Cloudy.
24	9	37	50	30,27	71	W	1	Cloudy.
	4	45	50	30,26	68	NW	1	Hazy.
25	8,30	43	49	30,35	73	N	1	Fine.
	3,30	48	52	30,38	71	W	1	Rain.
26	8,45	43	50	30,40	68	W	1	Foggy.
	3,30	47	54	30,49	68	W	1	Cloudy.
27	8,45	42	52	30,43	68	W	1	Foggy.
	4	46	54	30,41	68	SW	1	Cloudy.
28	9	43	52	30,41	71	W	1	Cloudy and thick.
	4	47	54	30,35	72	W	1	Cloudy.
29	9	42	52	30,33	72	W	1	Cloudy.
	4	46	54	30,28	73	NW	1	Cloudy.
30	9	46	51	30,25	72	NW	1	Cloudy.
		51	58	30,20	72	NW	1	Cloudy.

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for December, 1811.

1811	Time.	Therm. without.	Therm. within.	Barom.	Hygrometer.	Winds.		Weather.
	H. M.	°	°	Inches.	°	Points.	Str.	
Dec. 1	10	48	54	30,16	73	SW	1	Cloudy.
	4	49	55	30,00	70	SSW	1	Cloudy.
2	9	48	53	29,57	72	W	1	Rain.
	3,45	45	55	29,71	68	W	1	Fair.
3	8,45	43	52	29,94	69	W	1	Cloudy.
	3,15	48	55	29,78	67	WSW	1	Cloudy.
4	8,30	41	53	29,38	68	W by S	1	Fine.
	3,15	43	55	29,29	65	W	1	Cloudy.
5	8,45	34	51	29,48	64	NE	1	Cloudy.
	4	31	53	29,75	56	WNW	1	Fair.
6	9	29	50	30,01	66	W	1	Hazy.
	3	36	51	30,03	66	W	1	Hazy.
7	9	47	51	29,79	75	W	1	Rain.
	4,45	49	54	29,64	75	W	1	Rain.
8	9	51	53	29,47	77	S	1	Rain.
	3,40	50	53	29,36	72	S	2	Rain.
9	9	42	51	29,77	71	SSE	1	Cloudy.
	3,30	43	54	29,06	70	SW	1	Cloudy.
10	9	38	51	29,00	73	W	3	Cloudy. A violent squall in the night.
		43	54	29,27	70	W	1	Foggy.
11	8,45	39	51	29,69	70	N	1	Cloudy.
	11	38	53	29,94	75	NW	1	Cloudy.
12	9	35	50	29,98	74	W	1	Cloudy.
	3,45	42	54	29,92	75	W	1	Cloudy.
13	9	50	53	29,86	76	W	1	Cloudy.
	2,30	53	56	29,80	73	W	1	Cloudy.
14	9	38	53	29,85	66	WNW	1	Fine.
	3,10	41	55	29,94	63	WNW	1	Fine.
15	10	37	52	30,04	72	W	1	Cloudy.
	4	43	52	29,87	71	SW	1	Rain.
16	8,45	37	50	29,40	69	W	1	Fine.
	3,15	41	53	29,32	63	WNW	1	Fair.

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for December, 1811.

1811	Time. H. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °	Winds. Points.	Winds. Str.	Weather.
Dec. 17	9	39	50	29,59	68	W	1	Foggy.
	3,40	41	51	29,58	68	W	1	Cloudy.
18	9,15	36	49	29,91	72	WSW	1	Fine, somewhat hazy.
	3,40	42	51	29,88	70	S	1	Cloudy.
19	9	49	51	29,69	76	W	1	Cloudy.
	2,40	50	52	29,71	68	SW	1	Cloudy.
20	9	50	54	29,69	77	W	1	Rain.
	3,15	50	51	29,75	76	SW	1	Rain.
21	9	49	53	29,55	68	W	2	Cloudy. Much rain and wind in the night.
	2,50	46	54	29,79	70	N	1	Cloudy.
22	9,15	32	52	30,23	64	NW	1	Fine.
	3,30	36	51	30,24	62	WNW	1	Fine.
23	9	43	50	30,08	73	W	1	Foggy.
	3,30	48	53	30,08	69	WNW	1	Fine.
24	9	43	51	30,19	71	W	1	Foggy.
	3,15	44	53	30,16	72	W	1	Rain.
25	9,30	33	51	30,24	67	N	1	Fine, but rather hazy.
	3,40	37	48	30,25	66	N	1	Fine.
26	9	32	46	29,99	68	E	1	Foggy.
	3,30	34	49	29,82	70	NNE	1	Snow.
27	9	26	46	29,44	69	W	1	Hazy and thick.
	3,45	34	48	29,03	71	W	1	Rain.
28	9,15	33	46	29,16	73	E	1	Cloudy.
	3,45	34	49	29,32	72	N	1	Cloudy and snow.
29	10	31	45	29,62	69	NW	1	Cloudy.
	6	32	45	29,73	71	N	1	Cloudy.
30	9,15	27	43	29,97	71	N	1	Fine.
	4	32	47	29,99	70	N	1	Cloudy.
31	9	26	43	30,09	72	W	1	Thick and hazy.
	3,50	31	45	30,02	71	W	1	Cloudy.

1811.	Thermometer without.			Thermometer within.			Barometer.*			Hygrometer.		
	Greatest height.	Least height.	Mean height.	Greatest height.	Least height	Mean height.	Greatest height.	Least height.	Mean height.	Greatest height.	Least height.	Mean height.
	Deg	Deg.	Deg.	Deg.	Deg.	Deg.	Inches.	Inches.	Inches.	Deg.	Deg.	Deg.
January	45	24	34,9	51	36	44,9	30,51	29,09	29,88	78	58	68,4
February	50	32	42,5	55	46	51,4	30,22	28,94	29,53	77	60	68,5
March	57	39	46,7	64	46	55,3	30,56	29,42	30,10	79	50	65,3
April	67	37	52,1	63	51	57,5	30,39	29,12	29,74	76	50	63,3
May	72	48	60,1	69	57	61,8	30,04	29,41	29,76	75	51	63,4
June	71	55	62,2	68	60	63,3	30,28	29,11	29,86	76	51	59,7
July	73	56	64,9	72	61	68,0	30,23	29,67	29,99	75	53	62,4
August	71	55	62,5	68	59	64,1	30,33	29,46	29,92	70	53	61,2
September	72	45	61,0	73	55	64,9	30,31	29,26	29,99	72	51	62,7
October	65	45	57,6	67	52	60,4	30,28	28,71	29,67	74	61	68,5
November	61	32	47,3	60	49	51,2	30,49	29,36	29,99	76	62	69,1
December	53	26	40,2	56	43	51,1	30,25	29,00	29,59	77	62	70,0
Whole year			52,7			57,8			29,83			65,2

* The quicksilver in the bason of the barometer, is 81 feet above the level of low water spring tides at Somerset-house.

Rain from 25th May, 1811, to 31st December, 1811, 10,645 Inches.

Magnetic Needle, September, 1811. { Variation 24° 14′ 2″ West.
{ Dip - 70° 32′ 30″.

PHILOSOPHICAL

TRANSACTIONS,

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXII.

PART II.

LONDON,

PRINTED BY W. BULMER AND CO. CLEVELAND-ROW, ST. JAMES'S;
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MDCCCXII.

CONTENTS.

ERRATA.

Of the figures intended to illustrate Mr. PLAYFAIR's paper in the last volume of the Transactions, Fig. 3 is named by mistake Fig. 1. Fig. 1 was omitted by accident, and is to be found at the end of this Part of the Transactions.

Philosophical Transactions, 1811, page 230, line 13, *for* $8^h\ 35'\ 45''$ *read* $8^h\ 2'\ 5''$.

Page 151 of this volume of the Transactions, line 20, *for* half *read* double.

XII. *Observations of a second Comet, with Remarks on its Construction.* *By* William Herschel, *LL. D. F. R. S.*

Read March 12, 1812.

As we have lately had two comets to observe at the same time, I have called that of which the following observations are given, the second. Its appearance has been so totally different from that of the first, that every particular relating to its construction becomes valuable; and notwithstanding the unfavourable state of the weather at this time of the year, I have been sufficiently successful to obtain a few good views of the phenomena which this comet has afforded.

A short detail of the observations, in the order of their relation to the different cometic appearances, is as follows:

The Body of the Comet.

January 1, 1812. I viewed the second comet with several of my telescopes, and found it to have a considerable nucleus surrounded with very faint chevelure.

Jan. 2. The comet had a large round nucleus within its faint nebulosity. Not seeing it very well defined, and of so

large a diameter, I doubted whether it could be the body of the comet; but although it might be called very large when supposed to be of a planetary construction, it was much too small for the condensed light of a head; its diameter, by estimation not exceeding 5 or 6 seconds.

By way of comparing the two comets together I viewed them alternately. The first, within a nebulosity which in the form of a brilliant head was of great extent, had nothing resembling a nucleus: the light of this head was very gradually much brighter up to the very middle; its small planetary body being invisible. The second comet, on the contrary, although surrounded by a faint chevelure, seemed to be all nucleus; for the abrupt transition from the central light to that of the chevelure would not admit of the idea of a gradual condensation of nebulosity, such as I saw in the head of the first comet; but plainly pointed out that the nucleus and its chevelure were two distinct objects.

Jan. 8. The comet had a pretty well defined nucleus with very faint chevelure. When magnified 170 times the nucleus, though less bright, was rather better defined.

Jan. 18. Within a very faint chevelure I saw the nucleus as before.

Jan. 20. The air being uncommonly clear, I saw the body of the comet well defined; and as the moon was already so far advanced in its orbit as to render future opportunities of viewing the comet very improbable, I ascertained the magnitude of its body, with a very distinct 10 feet reflector, by the following three observations.

First with a low power, which gave a bright image of the nucleus, I kept my attention fixed upon its apparent size; then

looking away from the telescope, I mentally reviewed the impression its appearance had made on the imagination, in order to see whether it was a faithful picture of the object; and by looking again into the telescope I was satisfied of the similitude.

In the next place I used a deeper magnifier, and alternately viewed and remembered the appearance of the nucleus. It was fainter with this power.

The third observation was made in the same manner with a magnifier of 170. This showed the nucleus of a larger diameter, but much less bright, and not so well defined.

The next morning, having recourse to my usual experiment with a set of globules, by viewing them at a given distance with the same telescope and eye-glasses, I found that one of them, on which I fixed, gave me, as nearly as could be estimated, the same magnitude with the first eye-glass, and was proportionally magnified by the second and third, with only this difference, that the highest power showed the globule with more distinctness than it did the nucleus; and by trigonometry the angle under which I saw the globule was found to be 5″,2744.*

It will be necessary to mention that in the calculations belonging to this comet, I have used the elements of Mr. GAUS, with a small correction of the longitude of the perihelion, which I found would answer the end of giving the observed place with sufficient accuracy from the 1st of January to the

* I prefer this method of ascertaining the small diameter of a faint object to measuring it with a micrometer, which requires light to show the wires, and a high magnifying power to give an image sufficiently large for mensuration; neither of which conditions the present comet would admit.

20th. These calculations may however be repeated, if hereafter we should obtain elements improved by additional observations, made with fixed instruments; but the result, I may venture to say, will not be materially different.

The distance of the comet from the earth, the 20th of January when its apparent diameter was determined, was 1,0867, the mean distance of the earth from the sun being 1; whence we deduce a very remarkable consequence, which is, that the real diameter of its nucleus cannot be less than 2637 miles.

The Chevelure of the Comet.

Instead of that bright appearance, which in the first comet has been considered as the head, there was about the nucleus of the second a faint whitish scattered light, which may be called its chevelure.

Jan. 1. Examining the chevelure of the comet with a 10 feet reflector, I found that it surrounded the nucleus, not in the form of a head consisting of gradually much condensed nebulosity, but had the appearance of a faint haziness, which although of some extent, was not much brighter near the nucleus than at a distance from it.

Jan. 2. I viewed the two comets alternately. The first could only be distinguished from a bright globular nebula by the scattered light of its tail, which was still 2° 20′ long. The second comet, on the contrary, had nothing in its appearance resembling such a nebula: it consisted merely of a nucleus, surrounded by a very faint chevelure; and had it not been for an extremely faint light in a direction opposite to the sun, it would hardly have been intitled to the name of a comet;

having rather the appearance of a planet seen through an atmosphere full of haziness.

Jan. 8. The chevelure consisted of so faint a light that, when magnified only 170 times, it was nearly lost.

Jan. 18. The chevelure was extremely faint and of very little extent.

Jan. 20. The light of the moon, which was up, would not admit of further accurate observations on the chevelure.

The Tail of the Comet.

Jan. 1. With a low magnifying power, I saw in the 10 feet reflector an extremely faint scattered light, in opposition to the sun, forming the tail of the comet. It reached from the centre of the double eye-glass half way toward the circumference.

Jan. 8. The narrow, very faint scattered light beyond the chevelure remains extended in the direction opposite the sun.

Jan. 18. I estimated the length of the tail by the proportion it bore to the diameter of the field of the eye-glass, which takes in 38′ 39″, and found that it filled about one quarter of it, which gives 9′ 40″.

Jan. 20. On account of moonlight the tail was no longer visible.

From the angle which it subtended in the last observation, it will be found that its length must have been about 659 thousand miles.

Remarks on the Construction of the Comet.

The method I have taken in my last paper of comparing together the phenomena of different comets appears to me

most likely to throw some light upon a subject which still remains involved in great obscurity. When the comet of which the observations have been given in this paper is compared with the preceding one, it will be found to be extremely different. Its physical construction appears indeed to approach nearly to a planetary condition. In its magnitude it bears a considerable proportion to the size of the planets; the diameter of its nucleus being very nearly one-third of that of the earth.

The light by which we see it is probably also planetary; that is to say reflected from the sun. For were it of a phosphoric, self-luminous nature, we could hardly account for its little density: for instance, the very small body of the first comet, at the distance of 114 millions of miles from the earth bore a magnifying power of 600, and was even seen better with this than with a lower one;* whereas the second, notwithstanding its large size, and being only at the distance of 103 millions, had not light enough to bear conveniently to be magnified 107 times; but if we admit this nucleus to be opaque, like the bodies of the planets, and of a nature not to reflect much light, then its distance from the sun, which the 20th of January was above 174 millions of miles, will explain the cause of its feeble illumination.

That the nucleus of this comet was surrounded by an atmosphere appears from its chevelure, which, though faint, was of considerable extent; and the elasticity of this atmosphere may be inferred from the spherical figure of the chevelure, proved by its roundness and equal decrease of light at equal distances from the centre.

* See Observations of the First Comet.

The transparency of the atmosphere is partly ascertained from our seeing the nucleus through it, but may also be inferred by analogy from an observation of the first comet. It will be remembered that an atmosphere of great transparency, which had been seen for a long time, was lost when the comet receded from the sun, by the subsidence of some nebulous matter not sufficiently rarified to enter the regions of the tail.* Now as the existence of this atmosphere, when it was no longer visible, might have been doubted, the luminous matter suspended in it, which had already 20 days obstructed our view of it, happened fortunately to be once more elevated the 9th of December, and thereby enabled us, from its transparency and capacity of sustaining luminous vapours, to ascertain the continuance of its existence. By analogy, therefore, we may surmise that the faint chevelure of the second comet consists also of the condensation of some remaining phosphoric matter, suspended in the lower regions, of an elastic, transparent fluid, extending probably far beyond the chevelure without our being able to perceive it.

We might ascribe the little extent and extreme faintness of the tail to the great perihelion distance of the comet, if it had not already been proved, by the comparative view which in my last paper has been taken of the two comets of 1807 and 1811, that the effect of the solar agency depends entirely upon the state of the nebulous matter, which the comet in its approach exposes to the action of the sun. Our last comet therefore had probably but little *unperihelioned* matter in its atmosphere.

The high consolidation of the matter contained in the second

* See Observations of the First Comet.

comet is also much supported by the different appearance of the two comets in the observation of the 2d of January. In order to judge of them properly, we must consider their situation with regard to the sun and the earth; the first comet was 192 millions of miles from the sun; the second only 164: the first was at the same time 262 millions from the earth: the second only 83; but notwithstanding the great disadvantage of being 28 millions of miles farther from the sun, and about 179 millions farther from the earth, the first comet had the luminous appearance of a brilliant head accompanied by a tail 45 millions of miles in length; whereas the second comet, so advantageously situated, had only a very faint chevelure about its large but faint nucleus, with a still fainter tail, whose length has been shown not much to exceed half a million.

If then the effect of the action of the sun on the comets at the time of their perihelion passage is more or less conspicuous, according to the quantity of unperihelioned nebulous matter they contain, we may by observation of cometic phenomena arrange these celestial bodies into a certain order of consolidation, from which, in the end, a considerable insight into their nature and destination may be obtained. The three last observed comets, for instance, will give us already the following results.

The comet of which this paper contains observations, is of such a construction that it was but little more affected by a perihelion passage than a planet would have been. This may be ascribed to its very advanced state of consolidation, and to its having but a small share of phosphoric or nebulous matter in its construction.

That of the year 1807 was more affected, and although

considerably condensed, showed clearly that it conveyed a great quantity of nebulosity to the perihelion passage.

The comet of last year contained with little solidity a most abundant portion of nebulous matter, on which, in its approach to the perihelion, the action of the sun produced those beautiful phenomena, which have so favourably afforded an opportunity for critical observations.

XIII. *Additional Experiments on the Muriatic and Oxymuriatic Acids.* *By* William Henry, *M. D. F. R. S. V. P. of the Lit. and Phil. Society, and Physician to the Infirmary, at Manchester.*

Read March 19, 1812.

THE experiments, which form the subject of the following pages, are intended as supplementary to a more extensive series, which the Royal Society did me the honour to insert in their Transactions for the year 1800.* Of the general accuracy of those experiments, I have since had no reason to doubt; and their results, indeed, are coincident with those of subsequent writers of the highest authority in chemistry. My attention has been again drawn to the subject by the important controversy which has lately been carried on between Mr. MURRAY and Mr. JOHN DAVY, respecting the nature of muriatic and oxymuriatic acids;† and I have been induced, by some hints which the discussion has suggested, not only to repeat the principal experiments described in my memoir, but to institute others, with the advantage of a more perfect apparatus than I then possessed, and of greater experience in the management of these delicate processes.

This repetition of my former labours has discovered to me an instance, in which I have failed in drawing the proper conclusion from facts. In two comparative experiments on the electrization of equal quantities of muriatic acid gas, the one

* Page 188. † NICHOLSON's Journal, xxviii. and xxix.

of which was dried by muriate of lime, and the other was in its natural state, I found a difference of not more than one *per cent.* in the hydrogen evolved, relatively to the original bulk of the gas.* Yet, notwithstanding these results, I have expressed myself inclined to believe that some water is abstracted by that deliquescent salt; and this belief was confirmed, several years afterwards, by the event of an experiment in which muriatic acid gas, dried by muriate of lime, gave only $\frac{1}{35}$ its bulk of hydrogen,† a proportion much below the usual average. The question, however, was too interesting to be left in any degree of uncertainty; and I have, therefore, made several fresh experiments with the view to its decision. In the course of these I have found, that though differences in the results are produced by causes apparently trivial, some of which I shall afterwards point out, yet that under equal circumstances, precisely the same relative proportion of hydrogene gas is obtained from muriatic acid gas, whether exposed or not to muriate of lime; and that its greatest amount does not exceed $\frac{1}{16}$ or $\frac{1}{14}$ the original volume of the acid gas.

In the paper last quoted,‡ I have also described an experiment, in which sensible heat was evolved by bringing muriate of lime into contact with muriatic acid gas; a fact which, if established, would go far to prove the existence of water in that gas. But on repeating the experiment with muriate of lime recently cooled from fusion, and over mercury carefully deprived of all moisture by boiling, I was not able to discover any increase of temperature, though a very sensible air thermometer was inclosed in the vessel containing the gas. The evolution of heat takes place, only when the muriate of lime

* Page 191. † Phil. Trans. 1809, page 433. ‡ Page 433. note.

has attracted moisture, either from the atmosphere or the mercury, and is then owing to a condensation of a part of the gas.

Essentially, the changes produced by electrifying muriatic acid over mercury are those which I have stated; viz. a contraction of the volume of the gas, the formation of muriate of mercury (calomel), and the evolution of hydrogen. Recent experiments, also, have confirmed the accuracy of the observation,* that when a certain effect has been produced by electricity, nothing is gained by continuing the process; for neither is more hydrogen evolved, nor can the contraction of bulk be carried any farther.

I have lately applied, to experiments on muriatic acid, an apparatus which I used advantageously for the analysis of ammonia.† It consists of a spherical glass vessel, into which are hermetically sealed two small tubes containing platina wires, the points of which approach within the striking distance. To the globular part is attached a neck, which may be closed, as occasion requires, either by a glass stopper or by a metal cap and stop-cock. Into a vessel of this kind, I introduced $4\frac{1}{2}$ cubic inches of muriatic acid gas, and passed through it 3000 discharges from a Leyden jar; at the close of the process, no traces of moisture could be perceived on the inner surface of the vessel, nor could I discover, on opening the stopper, that any change of bulk had taken place. After absorbing the unchanged muriatic acid gas by a small quantity of water, a volume of gas remained, in which there were present 100 measures (each equal to one grain of mercury) of oxymuriatic acid gas, and 140 measures of hydrogen. Two

* Phil. Trans. 1800. p. 192. † Ibid. 1809.

causes might, perhaps, contribute to diminish, in some degree, the proportion of the former. It was difficult to exclude from the apparatus, on admitting the muriatic acid gas into it, two or three very minute globules of mercury, which became tarnished during the experiment, exactly as they would have been by oxymuriatic acid; and a small portion of the latter gas was probably also taken up by the water employed to absorb the muriatic acid.

With the intention of giving greater effect to the electricity, I repeated the experiment in a vessel capable of containing not more than 1400 grains of quicksilver (about .41 of a cubic inch), the neck of which, being only $\frac{1}{5}$ of an inch in diameter, was better calculated to shew any minute change in the volume of the gas. On removing the stopper, however, no change of volume was apparent. The hydrogen evolved, instead of being more than in the former experiment, equalled in bulk only 20 grains of mercury. The production of oxymuriatic acid was sufficiently evinced by its effect in tarnishing some very small globules of quicksilver, which adhered to the inside of the vessel; but the minuteness of the quantity frustrated an attempt to measure it. From subsequent experiments on similar quantities of gas, confined in the same apparatus, it appeared that the electrization in this last instance had been continued much longer than was necessary; and that an equal effect was produced by $\frac{1}{8}$ the number of electrical discharges.

In this way of making the experiment, the greatest proportion of hydrogen gas obtainable from muriatic acid amounted only to about $\frac{1}{70}$th, while, by electrization over quicksilver, $\frac{1}{16}$ or $\frac{1}{14}$ was generally evolved. It was evident, then, that

the mercury had considerable influence over the results; and I found, by experiments with tubes of different diameters, that the larger the surface of the mercury exposed to the gas, the more rapid and complete was the change. Its action was greatly accelerated, also, by causing the electric discharge to strike from the conducting wire, sealed into the tube, to the mercury, which was probably thus raised into vapour; for in some instances, the whole of the inner surface of the glass was coated with sublimed calomel.

The only way, in which the mercury appeared to me likely to be efficient in this case, was by removing the oxymuriatic acid as fast as it was formed; for I have never found any mixture of this gas in the results of experiments on muriatic acid, when carried on over quicksilver. Upon any theory of the constitution of muriatic acid, it may be expected that when, in a mixture of that acid gas with hydrogen and oxymuriatic acid gases, the two latter come to bear a certain proportion to the former, they will be brought within the sphere of mutual agency, and will reproduce muriatic acid. This point appears, from my experiments, to be attained, when the hydrogen and oxymuriatic acid, taken together, have the proportion to the muriatic acid, of about 1 to 35. The amount of the change, therefore, which is capable of being effected on muriatic acid gas, electrified without the contact of mercury, is limited by the reaction of the evolved hydrogen and oxymuriatic acid gases on each other, whenever they compose a certain proportion of the mixture. This proportion being attained, we only, by continuing the electrization, work in a circle.

It may now be inquired, what is the limitation to the action of electricity on muriatic acid gas which is confined over mer-

cury? In this case, it was suggested to me by Mr. DALTON, who favoured me with his presence at most of the experiments, that the evolved hydrogen might possibly in some way prevent the effect from being carried beyond a certain amount. Availing myself of this hint, I mixed 30 measures of hydrogen gas with 400 of muriatic acid gas in its ordinary state, and passed 900 discharges through the mixture. It soon became evident that the addition of the hydrogen had produced an important difference in the results of the experiment; for the surface of the mercury, over which the gas rested, was untarnished after some hundred explosions, and was scarcely changed at the close of the process. When the residuary gas, the volume of which remained unaltered, was analyzed, it was found to contain the same quantity of muriatic gas as at the outset, and neither more nor less hydrogen. To explain the event of this modification of the experiment, on the old theory, we may suppose that by the action of electricity a particle of water is decomposed, and that the atom of oxygen, forcibly repelled from that of hydrogen with which it was associated, finds another atom of hydrogen uninfluenced by the electric fluid, and within the sphere of its attraction. With this it unites, and recomposes water. On the theory of Sir H. DAVY, the same series of decompositions and recombinations may be assumed to take place between the oxymuriatic acid and hydrogen.*

* I am aware that there is an apparent inconsistency in supposing changes of precisely an opposite kind to be effected by the same means. But instances are not wanting, in which the very same elements are brought into combination by electric discharges, and are again disunited by the same agency. As examples, it may be sufficient at present to state, that nitrous acid and nitrous gas are generated by the action of the electric spark on mixtures of oxygen and nitrogen gases; and that, by

It still, however, remains to be determined, what is the source of the hydrogene gas, which, in a limited proportion, is always evolved by the electrization of muriatic acid? Does it result from the decomposition of water, existing as an element of the gas; or from the disunion of the oxymuriatic acid and hydrogen, which, according to Sir H. DAVY's view, compose muriatic acid? The limitation to its amount, which, it formerly appeared to me,* could only be accounted for by the complete destruction of the water contained in the gas, may now be equally well explained, on the principle which I have just pointed out. The fact, also, that no appreciable change of bulk is produced by the electrization of the muriatic acid, when the presence of mercury is excluded, is perhaps favourable to the new theory. For since equal measures of hydrogen and oxymuriatic acids afford muriatic acid without any condensation of volume, no alteration of bulk should result from the disunion of those elements; and the products should be equal measures of the same gases. The proportions, which I obtained (100 to 140) did not, it must be acknowledged, exactly correspond with the theory; but the difference was not greater, than might naturally be expected from the circumstances of the experiment. That equal measures of hydrogen and oxymuriatic acid are really evolved, appears to me to be proved by the agreement, which I have in several experiments remarked, between the hydrogen gas obtained,

the same power, they are again resolved into their elements. If this were the proper place, it might, I think, be rendered probable by several arguments, that electricity, when thus applied, acts rather by mechanical collision, than by inducing a change in the electrical states of the elements of bodies.

* Phil. Trans. 1800, p. 200.

and the contraction of volume in muriatic acid electrified over mercury. Now the latter effect of the process can be explained on no other principle than the absorption of oxymuriatic acid by the quicksilver.

When muriatic acid and oxygen gases are electrified together over mercury, a gradual diminution ensues in their bulk,* and the mercury becomes tarnished, precisely as by the contact of oxymuriatic acid. I have lately examined the agency of this process on a considerable quantity of the two gases confined in a vessel, into which they were admitted after exhausting it by the air-pump. The phenomena, which in this way of making the experiment are extremely decisive and interesting, are the production of water and of oxymuriatic acid. The former, combining with a portion of the undecomposed muriatic acid, is deposited in drops upon the inner surface of the vessel, in the state of liquid muriatic acid. When the stop-cock, which confine the gases, is opened under mercury, a quantity of that metal rushes in, and has its surface instantly tarnished. Besides this test of the production of oxymuriatic acid, its presence is rendered unequivocal (after absorbing the undecomposed muriatic acid by a few drops of water), both by its smell, and by its effect in discharging the colour of litmus paper.†

These results, it will be found, may be reconciled with

* Phil. Trans. 1800, p. 193.

† Those who wish to repeat this experiment need not be deterred by the apprehension of the labour attending it; for 3 or 400 discharges, from a Leyden jar of moderate size, are sufficient to occasion a distinct precipitation of moisture. When a mixture of oxygen and muriatic acid gases is even suffered to stand over mercury, a gradual contraction of volume takes place; the muriatic acid, if in proper proportion, entirely disappears; and calomel is deposited upon the surface of the glass vessel; but, in this case, there is no visible production of moisture.

either theory. According to the one which has been commonly received, the oxygen unites with the real acid of muriatic gas, which becoming oxymuriatic acid, *deposits* water. On Sir H. DAVY's view, the oxygen unites with the hydrogen of the muriatic acid, and *composes* water, while the oxymuriatic acid is merely an educt. I am not aware of any refinement of the process, by which the value of these two explanations can be compared. Something, however, would be gained by a precise determination of the proportions, in which the two gases saturate each other. For since, on Sir H. DAVY's theory, muriatic acid contains half its volume of hydrogen gas, two measures of which are known to be saturated by one of oxygen, it follows that muriatic acid gas should be changed into oxymuriatic by one-fourth of its bulk of oxygen. According to GAY LUSSAC and THENARD,* three measures of muriatic acid should condense one of oxygen (or only one-third their bulk), and should form two measures of oxymuriatic acid. Hitherto, I have not been able to satisfy myself respecting the true proportions of oxygen and muriatic acid gases, that are capable of being united by electricity; for though I have made several experiments with this view, they have not agreed in yielding similar results. The condensation of a part of the undecomposed acid by the water, which is formed during the process, will, probably, indeed, always be an impediment to our learning these proportions exactly. The fact is chiefly of value, as it affords an example of the production of oxymuriatic acid under the simplest possible circumstances; and as it shews unequivocally that, under such circumstances, the visible appearance of moisture is a part of the phenomena.

* Mémoires d'Arcueil, ii. 217.

Manchester, Jan. 6, 1812.

XVI. *Of the Attraction of such Solids as are terminated by Planes; and of Solids of greatest Attraction.* *By* Thomas Knight, *Esq. Communicated by Sir* H. Davy, *LL.D. Sec. R. S.*

Read March 19, 1812.

MATHEMATICIANS, in treating of the attraction of bodies, have confined their attention, almost entirely, to those solids which are bounded by continuous curve surfaces; and Mr. PLAYFAIR, if I do not mistake, is the only writer, who has given any example of that kind of inquiry, which is the chief object of the present paper. This learned mathematician has found expressions* for the action of a parallelopiped; and of an isosceles pyramid, with a rectangular base, on a point at its vertex; and observes, on occasion of the first mentioned problem, that what he has there done, "gives some hopes of "being able to determine generally the attraction of solids "bounded by any planes whatever."

It is this general problem, that I venture to attempt the solution of, in what follows: viz. *any solid, regular or irregular, terminated by plane surfaces, being given, to find, both in quantity and direction, its action, on a point, given in position, either within or without it.*

* Ed. Trans. Vol. VI. p. 228 to 243. It is proper however to observe, that Mr. PLAYFAIR's expression, at p. 242, for the action of a parallelopiped, requires to have its sign changed; being, as it stands at present, negative, from the manner of correcting the fluent.

Nor has the matter any difficulty, as far as *theory** *only* is concerned; although, *actually* to find the attraction, of a body of very complicated figure, may, no doubt, be exceedingly laborious and troublesome: for no one, I suppose, will conceive, that it can be done in any other manner, than by a previous partition into more simple forms, each of which must have its action found separately.

Having completed this part of my subject in the three first sections, I next apply the formulas, given in §. 1, to find the attraction of certain complex bodies, which, though not bounded by planes, have yet a natural connexion with the preceding part of the paper. Finally, the fifth section treats, pretty fully, of solids of greatest attraction, under various circumstances; and I do not know, that any one of the problems there given has been before considered by mathematicians; whilst, on the other hand, the results of former writers are easily derived as corollaries.

For the sake of perspicuity, I have divided the paper into propositions, and shall terminate this short introduction by expressing a hope, that I may not be chargeable with unnecessary prolixity.

§. I.

Of the Attraction of Planes bounded by right Lines.

As all such figures may be divided into triangles, it seems natural to begin with these.

* It is usual, I think, with mathematicians, to consider a thing as done, when it can be pointed out how it *may be* done. Thus M. LAGRANGE, in his excellent work "*De la Résolution des Equations numériques,*" says (p. 43) "cette méthode ne laisse, "ce me semble, rien à desirer." where, of course, he can only mean, as far as relates to *theory*.

Prop. 1.

Let rvm, fig. 1, be a triangle, right angled at r, and pm a right line, perpendicular to the plane of the triangle, at the angular point m; it is required to find the attraction of the triangle, on the point p, both in quantity and direction.

Conceive a plane to pass through the point p, parallel to the plane of the triangle, and, in it, the lines pg, po, respectively parallel to rm, rv. The problem will be solved, if we find the actions of the triangle, in the directions of the three rectangular co-ordinates pm, pg, po.

Draw ks parallel to rv, and put $a =$ pm, $b =$ rm, $T =$ mk, $t =$ kq; then pq $= \sqrt{a^2 + T^2 + t^2}$. Let $r =$ tang. vmr, then ks $= r \times$ km.

The element of the plane at q is $\dot{T} \times \dot{t}$, and its action, on p, in the direction pq, is $\frac{\dot{T}\dot{t}}{a^2 + T^2 + t^2}$; by resolving which, and putting A, B, C for the actions of the triangle, in the directions pm, pg, po, we get

$$A = \iint \frac{a\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{3}{2}}};\ B = \iint \frac{T\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{3}{2}}};\ C = \iint \frac{\dot{T}t\dot{t}}{(a^2+T^2+t^2)^{\frac{3}{2}}};$$

in all which expressions, we must first take the fluent, with respect to t, from $t = o$, to $t = rT$; and afterwards, with respect to T, from $T = o$, to $T = b$. To begin with A,— a first operation gives

$$A = \int \frac{arT\dot{T}}{(a^2+T^2)(a^2+(1+r^2)T^2)^{\frac{1}{2}}}$$

which, if we put $\beta^2 = 1 + r^2$, will be changed to

$$A = \int \frac{arT\dot{T}}{\beta(a^2+T^2)\left(\frac{a^2}{\beta^2}+T^2\right)^{\frac{1}{2}}}.$$

Put $z^2 = \frac{a^2}{\beta^2} + T^2$, whence $T^2 = z^2 - \frac{a^2}{\beta^2}$, $T\dot{T} = z\dot{z}$: by substituting these values we get

$$A = \int \frac{a r \dot{z}}{\beta \left(a^2 - \frac{a^2}{\beta^2} - a^2 + z^2\right)} = (\text{because } \beta^2 - 1 = r^2) \int \frac{a r \dot{z}}{\beta \left(\frac{a^2}{\beta^2} r^2 + z^2\right)},$$

which, if we multiply both numerator and denominator by $\frac{\beta^2}{a^2 r^2}$, becomes $\int \frac{\frac{\beta}{ar}\dot{z}}{1 + \frac{\beta^2}{a^2 r^2} z^2} = \text{arc}\left(\text{tang.} = \frac{\beta}{ar} z\right)$, and, by putting for z and β their values, we have at last

$$*A = \text{arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right).$$

In like manner, a first integration of B gives

$$B = \int \frac{r T^2 \dot{T}}{(a^2 + T^2)(a^2 + (1+r^2) T^2)^{\frac{1}{2}}} = \int \frac{r T^2 \dot{T}}{\sqrt{1+r^2}\,(a^2 + T^2)\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}}$$

$$= \frac{r}{\sqrt{1+r^2}} \int \left\{ \frac{\dot{T}}{\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}} - \frac{a^2 \dot{T}}{(a^2 + T^2)\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}} \right\}.$$

Put $T = a$ tang. ϖ, then $\dot{T} = a$ sect. $^2\varpi\dot{\varpi}$, $a^2 + T^2 = a^2$ sect. $^2\varpi$; by this means the last term under the sign of the fluent is changed to

$$\frac{\dot{\varpi}}{\left(\frac{1}{1+r^2} + \text{tang.}\,^2\varpi\right)^{\frac{1}{2}}} = -\frac{\sqrt{1+r^2}}{r} \times \frac{r\,\dot{\overline{\sin.\,\varpi}}}{(1 + r^2 \sin.\,^2\varpi)^{\frac{1}{2}}};$$

wherefore, observing that tang. $\varpi = \frac{T}{a}$, and consequently sin. $\varpi = \frac{T}{\sqrt{a^2 + T^2}}$, we find at last

* This quantity can be put under another form, which may be better in some cases. If we denote by b' the side rv of the triangle, $r = \frac{b'}{b}$, and

$$A = \text{arc}\left(\text{tang.} = \frac{b}{b'} \times \frac{\sqrt{a^2 + b^2 + b'^2}}{a}\right) - \text{arc}\left(\text{tang.} = \frac{b}{b'}\right).$$

$$B = \frac{r}{\sqrt{1+r^2}} L\left(\frac{b}{a} + \sqrt{\frac{1}{1+r^2} + \frac{b^2}{a^2}}\right) - L\left\{\frac{rb}{\sqrt{a^2+b^2}} + \sqrt{1 + \frac{r^2 b^2}{a^2+b^2}}\right\}$$
$$- \frac{r}{\sqrt{1+r^2}} L . \frac{1}{\sqrt{1+r^2}}.$$

We have yet to find the value of C; and at first we get

$$C = \int \frac{\dot{T}}{(a^2 + T^2)^{\frac{1}{2}}} - \int \frac{\dot{T}}{(a^2 + (1+r^2)T^2)^{\frac{1}{2}}} = \int \frac{\frac{\dot{T}}{a}}{\left(1 + \frac{T^2}{a^2}\right)^{\frac{1}{2}}} - \frac{\dot{T}}{\sqrt{1+r^2}}$$
$$\int \frac{\frac{\dot{T}}{a}}{\left(\frac{1}{1+r^2} + \frac{T^2}{a^2}\right)^{\frac{1}{2}}},$$

which again integrated, becomes

$$C = L\left(\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}}\right) - \frac{1}{\sqrt{1+r^2}} L\left(\frac{b}{a} + \sqrt{\frac{1}{1+r^2} + \frac{b^2}{a^2}}\right)$$
$$+ \frac{1}{\sqrt{1+r^2}} L . \frac{1}{\sqrt{1+r^2}}.$$

The expressions we have thus arrived at, for the action of a right angled triangle, are of such continual use in the following propositions, that it will be convenient to represent them by some concise symbol; and as they are functions of a, b, and r we may put

$$A = \text{arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) =$$
$$\phi(a, b, r) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right).$$

$$B = \frac{r}{\sqrt{1+r^2}} L\left(\frac{\sqrt{1+r^2}\, b}{a} + \sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) - L\left(\frac{rb + \sqrt{a^2 + (1+r^2)b^2}}{\sqrt{a^2+b^2}}\right)$$
$$= \chi(a, b, r).$$

$$C = L\left(\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}}\right) - \frac{1}{\sqrt{1+r^2}} L\left(\frac{\sqrt{1+r^2}\, b}{a} + \sqrt{1 + \frac{1+r^2\, b^2}{a^2}}\right)$$
$$= \psi(a, b, r).$$

Cor. 1. If, whilst r remains constant, b and a are supposed to vary, but so as to preserve the same ratio to each other, the

partial forces A, B, C will remain unchanged, as, consequently, will the total force, both in quantity and direction.

For, if we put $m \times a$ for b, the expressions of the forces become $\varphi(1, m, r) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right), \chi(1, m, r), \psi(1, m, r)$; which are independent of the *absolute* values of a and b. It is scarcely necessary to observe, that arc $\left(\text{tang.} = \frac{1}{r}\right)$ is the arc, to the radius unity, corresponding to the angle rvm.

Cor. 2. When r becomes infinite, the triangle rmv is changed into a parallelogram, infinitely extended in the direction rv; in which case, the expressions of the forces become very simple, viz. $A = \text{arc}\left(\text{tang.} = \frac{b}{a}\right)$, $B = L. \frac{\sqrt{a^2+b^2}}{a}$, $C = L. \frac{b+\sqrt{a^2+b^2}}{a}$.

Prop. 2.

Let vmu, fig. 2, be any triangle whatever, pm a line perpendicular to the plane of the triangle, at the angular point m: from whence, let fall the perpendicular mr on the opposite side uv; moreover, let pg, po, be respectively parallel to mr, vu.

It is required to find the actions of the triangle vmu on the point p, in the directions pm, pg, po.

If we keep the same denominations as before, and put, besides, $r' =$ tang. umr, it is plain from the last proposition, and because the action of the whole must necessarily equal the sum of the actions of its parts, that

$A = \varphi(a,b,r) + \varphi(a,b,r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r'}\right)$;
$B = \chi(a,b,r) + \chi(a,b,r')$; $C = \psi(a,b,r) - (a,b,r')$.

When umv is a right angle, we shall evidently have arc

$\left(\text{tang.} = \frac{1}{r}\right) + \text{arc}\left(\text{tang.} = \frac{1}{r'}\right) = \frac{\pi}{2}$, π being the number 3,1415, &c.: this makes the expression for A somewhat simpler, in that case.

If it is the triangle vmu′ whose attraction we seek, we have, putting $r' =$ tang. u′mr,

$$A = \varphi(a,b,r) - \varphi(a,b,r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) + \text{arc}\left(\text{tang.} = \frac{1}{r'}\right);$$

$$B = \chi(a,b,r) - \chi(a,b,r'); \; C = \psi(a,b,r) - \psi(a,b,r').$$

Cor. 1. As a rhombus may be divided, from its centre, into four equal triangles, like that in fig. 2, but right angled at m, the angle lying at the centre; if b represent a perpendicular from the centre of a rhombus on one of its sides, and r and r' the tangents of the angles, that this perpendicular makes, at the centre, with the semi-diameters of the figure, we shall have for the action of the rhombus, on a point situated perpendicularly over its centre, at the distance a,

$$A = 4\varphi(a,b,r) + 4\varphi(a,b,r') - 2\pi.$$

Cor. 2. As any plane, terminated by right lines, may be divided into triangles from a point within it, we may find, by means of this proposition, the attraction of such a plane, on a point above it, both in quantity and direction. Let, for example, uvu′v′u, fig. 6, be the plane, p the attracted point; let fall the perpendicular pm on the plane, and from m draw right lines to the angles u, v, u′, v′; the plane will thus be divided into triangles, situated, with respect to the point p, like that in the proposition.

The attraction may still be found, if the perpendicular should fall without the figure; as in

Prop. 3.

To find the attraction of a triangle umv, fig. 3, on a point p any how situated.

Let fall from p the perpendicular pm′, on the plane of the triangle; join m′m, m′u, m′v. Find, by the last Prop. the attractions of the triangles m′uv, m′um, m′vm, on the point p; and resolve them into others in the directions of any three rectangular co-ordinates: and, when thus resolved, let the actions of

$$\left\{\begin{matrix} m'uv \\ m'um \\ m'vm \end{matrix}\right\} \text{ in the directions of these co-ordinates be } \left\{\begin{matrix} A\ ,\ B\ ,\ C \\ A',\ B',\ C' \\ A'',\ B'',\ C'' \end{matrix}\right\}.$$

It is plain, that the actions of the triangle umv, on p, in the directions of the same co-ordinates, will be

$$A - A' - A''; \ B - B' - B''; \ C - C' - C''.$$

There may be other cases of this proposition, in which the triangle and point are placed differently, with respect to each other, from what I have represented in fig. 3; but the reader, who understands the case that has been considered, will have no difficulty in any other that may occur.

Though the preceding propositions contain every thing that is necessary, for finding the attraction, both in quantity and direction, of any plane bounded by right lines; yet there are some cases worthy of a particular notice: as

Prop. 4.

To find the attraction of a rectangle mrvr′, fig. 4, on a point p situated perpendicularly over one of its corners as m.

Draw pg, po, parallel to mr, mr′, the sides of the rectangle; put $b =$ rm, $b' =$ r′m, $r =$ tang. rmv, $r' =$ tang. r′mv: then, if

A, B, C represent the actions of the rectangle, in the directions pm, pg, po, we find, by means of Proposition 1,

$$A = \varphi(a, b, r) + \varphi(a, b', r') - \frac{\pi}{2};$$

$$B = \chi(a, b, r) + \psi(a, b', r'); \quad C = \psi(a, b, r) + \chi(a, b', r').$$

We may eliminate r and r', from these expressions, by means of their values $\frac{b'}{b}$ and $\frac{b}{b'}$; and thus we may put A under a very simple form. It becomes, at first,

$A = \text{arc}\left(\text{tang.} = \frac{b}{b'} \times \frac{\sqrt{a^2+b^2+b'^2}}{a}\right) + \text{arc}\left(\text{tang.} = \frac{b'}{b} \times \frac{\sqrt{a^2+b^2+b'^2}}{a}\right) - \frac{\pi}{2}.$

But, by trigonometry, α and β representing any angles whatever, $\text{tang.}(\alpha + \beta = \frac{\text{tang.}\,\alpha + \text{tang.}\,\beta}{1 - \text{tang.}\,\alpha \times \text{tang.}\,\beta}$: the application of which formula gives us, instead of the foregoing expression,

$$A = \text{arc}\left(\text{tang.} = \frac{-a}{bb'}\sqrt{a^2 + b^2 + b'^2}\right) - \frac{\pi}{2};$$

and this again is easily changed into the following form,

$A = \text{arc tang.} = \frac{bb'}{a\sqrt{a^2+b^2+b'^2}}\Big)$, which is easily perceived to be the same as Mr. PLAYFAIR's expression.

In a similar manner, might the expressions for B and C be simplified: but it is perhaps easier to find new forms, *ab initio*. Thus we may get B immediately, from the double integral $B = \iint \frac{T\dot{i}\dot{t}}{(a^2+T^2+t^2)^{\frac{3}{2}}}$, if the fluent, with respect to t, be taken from $t = o$ to $t = b'$; and, with respect to T, from $T = o$ to $T = b$. The first integration gives $B = \int \frac{b'T\dot{T}}{(a^2+T^2)(a^2+b'^2+T^2)^{\frac{1}{2}}}$; and, by the second,

$$B = L.\frac{\sqrt{a^2+b^2}+b'}{a} - L.\frac{\sqrt{a^2+b^2+b'^2}+b'}{\sqrt{a^2+b^2}}.$$

It is plain that, to find C, we need only change b into b', and the reverse, in the last expression, whence

$$C = L.\frac{\sqrt{a^2+b^2}+b}{a} - L.\frac{\sqrt{a^2+b^2+b'^2}+b}{\sqrt{a^2+b'^2}}.$$

Prop. 5.

To find the attraction of a regular polygon, on a point situated perpendicularly over its centre.

As this figure is composed of isosceles triangles, if we put b for the perpendicular from the centre on one of the sides, and r for the tangent of half the angle at the centre, subtended by one of those sides, we have, by Prop. 1, for a polygon of n sides,

$$A = 2n \text{ arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}\right) - 2n \text{ arc}\left(\text{tang.} = \frac{1}{r}\right),$$

which, because the last term $= (n-2)\,\pi$, by Euclid 1.32. *Cor.* 1, is

$$A = 2n \text{ arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}\right) - (n-2)\,\pi.$$

Prop. 6.

To find the attraction of a circle, on a point situated perpendicularly over its centre.

This is only a particular case of the last proposition, when n is infinitely great and r infinitely small.

It is easy to see, that the arc whose tangent is $\frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}$ will have for its cosine $\frac{ra}{\sqrt{a^2+b^2}}$, if we keep only the first power of r; consequently we may put it under the form $\frac{\pi}{2}$ — arc $\left(\text{sine} = \frac{ra}{\sqrt{a^2+b^2}}\right) = .\frac{\pi}{2} - \frac{ra}{\sqrt{a^2+b^2}}$, very nearly; this multi-

plied by $2n$ is $n\pi - \frac{2nra}{\sqrt{a^2+b^2}} = n\pi - \frac{2\pi a}{\sqrt{a^2+b^2}}$ by putting π for nr. If we substitute this value in the expression for a regular polygon, it becomes $A = 2\pi \left\{1 - \frac{a}{\sqrt{a^2+b^2}}\right\}$, the well known expression of NEWTON.

§. II.

Of the Attraction of Pyramids, and generally of any Solids whatever that are bounded by Planes.

The simplest case of the attraction of such bodies as we are to consider, is that of a pyramid with the attracted point at the vertex: and it fortunately happens, that on this simple case the action of any body whatever may be made to depend; which is the reason of my placing the general problem in this section, though I afterwards treat separately of prisms.

This facility, in the case of pyramids, results from what was shewn in *Cor.* 1. *Prop.* 1, viz. that if we put x for a and mx for b, in the functions $\varphi\,(a,b,r)$, $\chi\,(a,b,r)$, $\psi\,(a,b,r)$, they will become $\varphi\,(1,m,r)$, $\chi\,(1,m,r)$ $\psi\,(1,m,r)$, into which x does not enter.

Prop. 7.

Let figure 5 represent a pyramid with a triangular base umv, the vertex p of the solid being in a line pm, perpendicular to the triangle at the angular point m. It is required to find the action of the pyramid on a point at that vertex.

Draw the perpendicular mr, also pg, po parallel to mr, rv. Join pr, and let r′m′ be parallel to rm. Call pm′, x; r′m′, y; then $y = mx$, m being the tangent of the angle rpm. The attraction of a triangular section of the solid, made by a plane,

passing through r′m′, parallel to the base, will be found by *Prop.* 2, if we put x for a, and mx for b; and is in the respective directions pm, pg, po.

$A = \varphi(1, m, r) + \psi(1, m, r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r'}\right)$;

$B = \chi(1, m, r) + \chi(1, m, r')$; $C = \psi(1, m, r) - \psi(1, m, r')$.

If we multiply these by $\dot{x}$ and take the fluents, the actions of the pyramid are found to be

$A = x\varphi(1, m, r) + x\varphi(1, m, r') - x\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right) - x\,\text{arc}\left(\text{tang.} = \frac{1}{r'}\right)$;

$B = x\chi(1, m, r) + x\chi(1, m, r')$; $C = x\psi(1, m, r) - x\psi(1, m, r')$.

If the pyramid, whose action we were seeking, had been that whose is u′mv, we must have used the other values of A, B, C given in *Prop.* 2.

Prop. 8.

Let fig. 6 represent any pyramid whatever, whose base uvu′v′u is terminated by right lines; to find its attraction, on a point at the vertex p, both in quantity and direction.

Let a perpendicular from p meet the base at m, and draw lines from this point to all the angles u, v, &c. of the base. It is plain, that the solid will thus be divided into such pyramids as were considered in the last proposition; so that the problem is already solved.

Cor. We may apply this to the pyramid whose base is a rhombus, and the vertex placed perpendicularly over its centre. By proceeding as in the proposition, it will be divided

into four equal triangular pyramids; and, using for each of them the notation of *Prop.* 7, we have for the action of the whole rhomboidal pyramid, on a point at its vertex,

$$A = 4x\varphi\,(1, m, r) + 4x\varphi\,(1, m, r') - 2\pi x.$$

The other attractions evidently destroy each other.

It is not necessary, in the above proposition, that the perpendicular pm should fall within the base; if it falls without, we shall however have occasion for the following problem.

Prop. 9.

Let umv, fig. 3, be any triangle whatever, p a point any how situated with respect to it; join pm, pu, pv.* It is required to find the attraction of the oblique pyramid pumv, whose base is the triangle umv, on a point at the vertex p.

Let fall, from p, the perpendicular pm′ on the plane of the base umv, draw the lines m′m, m′u, m′v. Find, by *Prop.* 7, the attractions of the pyramids pm′uv, pm′um, pm′vm, whose bases are m′uv, m′um, m′vm, and their common vertex p.

Resolve these attractions into others in the directions of any rectangular co-ordinates, and when thus resolved let the actions of the pyramids

$$\left\{\begin{matrix}\text{pm}'\text{uv}\\ \text{pm}'\text{um}\\ \text{pm}'\text{vm}\end{matrix}\right\}\ \text{in the directions of these co-ordinates be}\ \left\{\begin{matrix}A, & B, & C\\ A', & B', & C'\\ A'', & B'', & C''\end{matrix}\right\}.$$

It is plain, that the actions of the pyramid pumv, on the point p, in the directions of the same co-ordinates, will be $A - A' - A''$; $B - B' - B''$; $C - C' - C''$.

* I have not actually drawn the lines, to avoid confusion in the figure.

Prop. 10.

Let fig. 4 represent, in every respect, the same as it did in *Prop.* 4; join pr, pv, pr′; it is required to find the attraction of the square pyramid pmrvr′, on the point p at its vertex.

If we proceed as in *Prop.* 7, but make use of the expressions, for the action of the rectangle mrvr′, found in *Prop.* 4, and put x for a, mx for b, $m'x$ for b', (where $m =$ tang. rpm, m' $=$ tang. r′pm) there will result

$$A = x\phi\,(1, m, r) + x\phi\,(1, m', r') - \frac{\pi x}{2};$$

$B = x\chi\,(1, m, r) + x\psi\,(1, m', r')$; $C = x\psi\,(1, m, r) + x\chi\,(1, m', r')$.

But it will be better to make use of the more simple expressions, that were given in *Prop.* 4, by which means, we get

$$A = \text{arc}\left(\text{tang.} = \frac{mm'}{\sqrt{1+m^2+m'^2}}\right) x$$

$$B = \left\{L.\,(\sqrt{1+m'^2} + m') - L.\,\frac{\sqrt{1+m'^2+m^2} + m'}{\sqrt{1+m^2}}\right\} x$$

$$C = \left\{L.\,(\sqrt{1+m^2} + m) - L.\,\frac{\sqrt{1+m^2+m'^2} + m}{\sqrt{1+m'^2}}\right\} x.$$

Prop. 11.

Let the base of the pyramid be a regular polygon; the vertex situated perpendicularly over the centre of the base; the attracted point at the vertex.

By making use of the expression in Proposition 5, putting x for a, and mx for b (m being the tangent of the angle, at the vertex, formed by the axis of the pyramid and a line drawn from the vertex to the middle of one of the sides of the base, we get

$A = 2nx \text{ arc } (\text{tang.} = \frac{1}{r}\sqrt{1 + (1+r^2)m^2}) - (n-2)\pi x.$

Hitherto, I have considered the action of a pyramid, only on a point at its vertex; the case which next presents itself, is that of the attraction on a point p′ (fig. 5 and 6) any where in the produced axis mpp′. It would be easy to give a direct solution to this problem, but I choose rather to make it depend on the propositions that have been already established: and to shew that the functions $\phi\ (a, b, r)\ \chi\ (a, b, r)\ \psi\ (a, b, r)$, of which so much use has been made, in the preceding investigations, are sufficient in all cases of the attraction of bodies bounded by planes.

Prop. 12.

Let pumv (fig. 5) represent the same pyramid as in *Prop.* 7; to find its attraction on a point p′ in the produced axis.

Suppose p′u, p′v joined, the attraction of the pyramid pumv, on the point p′, is the difference of the attractions of the pyramids p′umv, p′upv on the same point; which point being at the common vertex of these two pyramids, their attractions are found by Propositions 7 and 9; and the problem is solved.

Prop. 13.

Let it now be the action of the pyramid pumv, fig. 7, (where the plane of the base umv is not perpendicular to the line mpp′) on the point p′, that is required.

The attraction sought for will still be the difference of the actions of the two pyramids p′umv, p′upv, but these must now *both* be found by *Prop.* 9.

Prop. 14.

Figure 6 representing, in every respect, the same as in *Prop.* 8, it is required to find the action of the pyramid, on the point p′, in the produced axis.

As this solid may be divided into others, situated, with respect to p′, like that in *Prop.* 12, the problem is solved by what was shewn there.

If the attracted point was not in the line p′pm, perpendicular to the base, but in some other line πpμ, passing through the vertex, and meeting the base in μ; draw lines from μ to all the angles of the base, and the solid will be divided in such pyramids as were treated of in *Prop.* 13.

Prop. 15.

Let pumv, fig. 7, be any triangular pyramid whatever, and let it be any how cut by a plane, whose intersection with the pyramid is the triangle $\alpha\beta\gamma$; it is required to determine the action of the portion $\varphi\beta\gamma$vum (which is cut off by the plane) on a point at p.

The attraction, of the solid in question, is the difference of the actions of the pyramids pumv and p$\alpha\beta\gamma$, which actions are found by *Prop.* 9.

Prop. 16, *or general Problem.*

To find the action of any solid, bounded by planes, on a point either within or without it.

It is plain, that by drawing lines, from the attracted point through the solid, this may always be divided, either into such pyramids (with the point at the vertex) as were considered in

Prop. 9, or into portions of pyramids, like that treated of in *Prop.* 15; and consequently the solution of the problem may be obtained by means of these propositions.

§. III.

Of the Attraction of Prisms.

Prop. 17.

To find the attraction of a right prism, whose base is a regular polygon, on a point in the produced axis.

We saw, in *Prop.* 5, that the action of a regular polygon, on a point situated perpendicularly over its centre is

$$A = 2n \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2}b^2}\right) - (n-2)\pi.$$

To find the attraction of the prism, change a into x, multiply by $\dot{x}$, and take the fluent.

Now $\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)b^2}\right) = \int \dot{x}\,\phi(x,b,r)$ $= x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)b^2}\right) - \int x \, \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{1}{rx}\right.$ $\left.\sqrt{x^2 + (1+r^2)b^2}\right) = x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)b^2}\right)$ $+ \int \frac{rb^2 x\dot{x}}{(b^2+x^2)\sqrt{(1+r^2)b^2+x^2}}$, because $\dot{\overline{\text{arc}}} = \frac{\dot{\overline{\text{tang.}}}}{1+\text{tang.}^2}$; and, by taking the fluent,* it becomes $= x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\right.$ $\left.\sqrt{x^2 + (1+r^2)b^2}\right) - b\,\text{L}.\frac{rb + \sqrt{(1+r^2)b^2+x^2}}{\sqrt{b^2+x^2}}$.

* Put $x^2 + (1+r^2)b^2 = z^2$, $b^2 + x^2 = z^2 - r^2b^2$, $x\dot{x} = z\dot{z}$; then $\int \frac{rb^2 x\dot{x}}{(b^2+x^2)\sqrt{(1+r^2)b^2+x^2}} = \int \frac{rb^2\dot{z}}{z^2 - r^2b^2} = \frac{b}{2}\,\text{L}.\frac{z+rb}{z-rb}$ (Simpson's Fluxions, p. 140,) $= \frac{b}{2}\,\text{L}.\frac{(z+rb)^2}{z^2-r^2b^2} = b\,\text{L}.\frac{\sqrt{(1+r^2)b^2+x^2} + rb}{\sqrt{b^2+x^2}}$.

For the sake of brevity, call this quantity $F(b, r, x)$ and we have for the attraction of the prism,

$$A = 2nF(b, r, x) - (n-2)\pi x + \text{corr.}$$

The attraction of any other right prism, *in the direction of its length*, depends on the same function $F(b, r, x)$; as in

Prop. 18.

To find the attraction of a right prism, whose base is a rectangle, on a point in the produced axis.

We saw in *Prop.* 4, that the action of a rectangle, on a point situated perpendicularly over its centre is

$$A = 4\varphi(a, b, r) + 4\varphi(a, b', r') - 2\pi,$$

where b and b' are the halves of the sides of the rectangle, and r and r' the tangents of the angles, formed respectively by those sides and the diagonal. By changing a into x, and multiplying by $\dot{x}$, we have, for the prism,

$A = 4\int\dot{x}\varphi(x, b, r) + 4\int\dot{x}\varphi(x, b', r') - 2\pi x$; whence, by what was done in the last proposition,

$$A = 4F(b, r, x) + 4F(b', r', x) - 2\pi x + \text{corr.}$$

Prop. 19.

Let the base of the prism be a rhombus, the attracted point in the produced axis of the prism.

We found, in *Cor.* 1, *Prop.* 2, that the action of a rhombus on a point situated perpendicularly over its centre, is (keeping the notation there used)

$A = 4\varphi(a, b, r) + 4\varphi(a, b, r') - 2\pi$, therefore, proceeding as before, we have, for the prism,

$$A = 4F(b, r, x) + 4F(b, r', x) - 2\pi x + \text{corr.}$$

We will now consider the attractions of prisms more generally.

Prop. 20.

Let pr′v′vrm, fig. 8, be a right prism, whose base is the triangle vrm right angled at r. It is required to determine its attraction on a point at p, in the directions pm, pr′, po; po being parallel to r′v′.

If we wish to obtain a solution by means of what has been already done, we may conceive the solid under consideration to be divided into two pyramids; viz. the pyramid pmrv with the triangular base mrv; and the other pyramid pr′v′vr, whose base is the rectangle r′v′vr; the point p being at the common vertex of both.

Put pm $= x$, mr $=$ pr′ $= x'$; conceive the diagonal vr′ to be drawn, and put tang. rmv $=$ tang. r′pv′ $= r$; tang. v′rr $= r'$; tang. vr′v′ $= r''$; tang. rpm $= m$; tang. rpr′ $= m'$.

We get immediately, from propositions 7 and 10 (putting A, B, C for the respective actions in the directions pm, pr′, po)

$$A = x\phi\,(1, m, r) - x \text{ arc } \left(\text{tang.} = \frac{1}{r}\right) + x'\chi\,(1, m', r') + x'\psi\,(1, r, r'');$$

$$B = x\chi\,(1, m, r) + x'\phi\,(1, m', r') + x'\phi\,(1, r, r'') - x'\,\frac{\pi}{2};$$

$$C = x\psi\,(1, m, r) + x'\psi\,(1, m', r') + x'\chi\,)1, r, r'').$$

In finding these values, the first expressions of *Prop.* 10 were made use of; if we take the others, there will result the simpler forms

$$A = x\phi\,(1, m, r) - x \text{ arc } \left(\text{tang.} = \frac{1}{r}\right) + x'\left\{\text{L.}\,(\sqrt{1+r^2}+r) - \text{L.}\,\frac{\sqrt{1+r^2+m'^2}+r}{\sqrt{1+m'^2}}\right\};$$

$$B = x\chi\,(1, m, r) + x' \text{ arc } \left(\text{tang.} = \frac{rm'}{\sqrt{1+r^2+m'^2}}\right);$$

$C = x\psi\,(1, m, r) + x'\left\{L\,.\,(\sqrt{1+m'^2} + m') - L\,.\,\frac{\sqrt{1+m'^2+r^2} + m'}{\sqrt{1+r^2}}\right\}.$

These are, perhaps, as simple expressions as can be had in this case; we may however find others; nor is it necessary to conceive the solid divided into pyramids.

Thus we may obtain the value of the force A, by the method of the preceding propositions: a triangular section of the prism, parallel to the end r′pv′, and at a distance x from that end, will have its action on p expressed by $\varphi\,(x, x', r)$ — arc $\left(\text{tang.} = \frac{1}{r}\right)$, using the same notation as before; therefore, the action of the pyramid, in the direction pm, is

$A = \int \dot{x}\varphi\,(x, x', r) - x \text{ arc } \left(\text{tang.} = \frac{1}{r}\right) = F\,(x', r, x) - x \text{ arc } \left(\text{tang.} = \frac{1}{r}\right) + \text{corr.}$

Another Way of finding the force B.

Draw ks parallel to r′v′; call pk, x'; then ks $= r \times x'$. Conceive a plane to pass through ks, parallel to the back rvv′r′ of the prism; the section made by it will be a rectangle whose sides are x and rx'. The action of this rectangle on p, in the direction pr′, is arc $\left(\text{tang.} = \frac{rx}{\sqrt{x^2 + r^2x'^2 + x^2}}\right)$ by *Prop.* 4, wherefore the action of the prism, in the same direction, is

$B = \int \dot{x}' \text{ arc } \left(\text{tang.} = \frac{rx}{\sqrt{x^2 + (1+r^2)\,x^2}}\right)$ where x' is the variable quantity, we have then

$B = x' \text{ arc } \left(\text{tang.} = \frac{rx}{\sqrt{x^2 + (1+r^2)\,x'^2}}\right) - \int x' \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{rx}{\sqrt{x^2 + (1+r^2)x'^2}}\right).$

Put $\frac{x^2}{1+r^2} = c^2$, and the last term becomes $-\int \dot{x}' \overline{\text{arc}} \left(\text{tang.} \frac{rc}{\sqrt{c^2+x'^2}}\right) = + rc \int \left\{ \frac{\dot{x}'}{\sqrt{c^2+x'^2}} - \frac{(1+r^2)c^2\dot{x}'}{\left\{(1+r^2)c^2+x'^2\right\}\sqrt{c^2+x'^2}} \right\} = rc$
$\left\{ \mathrm{L}\left(x' + \sqrt{c^2+x'^2}\right) - \frac{\sqrt{1+r^2}}{r} \mathrm{L}\left(\frac{x'}{\sqrt{x^2+x'^2}} + \sqrt{\frac{1}{r^2} + \frac{x'^2}{x^2+x'^2}}\right) \right\}$;
therefore

$$B = x' \text{ arc } \left(\text{tang.} = \frac{rx}{\sqrt{x^2+(1+r^2)x'^2}}\right) + rc\,\mathrm{L}\left(x' + \sqrt{c^2+x'^2}\right)$$
$$- x\,\mathrm{L}\left(\frac{x'}{\sqrt{x^2+x'^2}} + \sqrt{\frac{1}{r^2} + \frac{x'^2}{x^2+x'^2}}\right) + \text{corr.}$$

If the fluent is to begin when $x' = o$, the correction is

$$- rc\,\mathrm{L}\,.\,c + x\,\mathrm{L}\,\frac{1}{r}.$$

I have dwelt the longer on this proposition, because the attraction of right prisms, in all cases, may be made to depend on it.

Cor. 1. It is in the first place evident, that by means of this proposition, we may find, by parts, the force, both in quantity and direction, with which a point q, any where on the edge pm of the scalene prism represented by fig. 9, is attracted.

The same may be said of the action on p′ any where in the produced axis; this will be the difference of the actions of two prisms, like that in the figure.

Cor. 2. Moreover, if, instead of the prism in fig. 9, the point q was placed any where on the edge of a prism whose base is the triangle quv, fig. 10, the action may still be found; for it will now depend on the difference of the action of such prisms as were treated of in the Proposition; that is to say, the action of the prism whose base is quv as the difference of the actions of those whose bases are qvr, qur, qr being a perpendicular on vu produced.

Prop. 21.

To find the quantity and direction of the attraction of a right prism, whose base is any triangle whatever, on a point any where on its surface.

Let the triangle uvu′, fig. 11, be a section of the prism, parallel to its base, and through the attracted point p. Let fall the perpendiculars pr, pr′, on the opposite sides: and the solid may be divided into four prisms, whose bases are the right angled triangles pur, prv, pvr′, pr′u′, and the attraction of each of these, both in quantity and direction, is given by *Prop.* 20.

It is plain that there may be other cases of this problem, besides the one here considered; for instance, one of the perpendiculars may fall beyond the base; but it would be endless, in a subject of this kind, to consider every particular case, and in none can the intelligent reader find the smallest difficulty.

Prop. 22.

To find the attraction of any prism, fig. 12, whose base is a *convex* polygon uvu′v′, on a point q any where within it.

As such a solid may be divided into triangular prisms, like those in *Prop.* 20 and its corollaries, with the attracted point on the common edge pm, the problem is already solved.

If the point be at p′, in the line mp produced, the action on it may still be found, being the difference of the actions of two prisms, like that in the figure.

Prop. 23.

Let vuv′, fig. 13, be the section of an isosceles prism; pv a line passing through the vertex v perpendicularly on the

middle of the base. It is required to find the action of the prism on any point p in the line pv.

This action is equal to the action of the prism whose base is the triangle v′pu (given by *Prop.* 20), less the actions of the prisms whose bases, or sections, are the triangles v′vp, uvp, which are found by *Cor.* 2, *Prop.* 20.

Scholium.

I considered here only an isosceles prism; but the solution is just the same, if the section of the prism is any triangle whatever, as v′uv, fig. 14, and the action on a point p, (situated in the line uv produced) is required. For the attraction wanted will be the difference of attractions of the two prisms whose bases are the triangles v′up, v′vp, and these are given by *Cor.* 2, *Prop* 20.

Suppose the base of the prism, whose attraction is required, to be the trapezium v′u$\alpha\beta$, fig. 14, the action of this on p, being the difference of the actions of the triangular prisms, whose bases are v′uv, $\beta\alpha$v, is found by the case just now considered.

In this manner, might cases be multiplied without end; but I think it is sufficiently plain, that by means of the preceding propositions and scholium, we may find the action of any prism whatever, on a point either within or without it.

§. IV.

Of the Attraction of certain Solids not terminated by Planes.

The expressions, arrived at in the first section, are useful in finding the attraction, not only of such solids as are bounded by planes, but of a great variety of others; viz. of such as

have their sections in one direction continuous curves, whilst, being cut in a different way, there results, from their intersection with a plane, a polygon, or rectangle, or some other right lined figure.

As no one, that I know of, has considered the actions of such bodies, I shall offer no apology for giving a few examples.

Let uvv′u′, fig. 15, represent any regular polygon, whose plane is perpendicular to the line pm, and its centre in that line: moreover, let this polygon be variable in magnitude, and move parallel to itself in the direction pm, in such a manner, that the middle point r of each of its sides uv, may describe a given curve pr.

Prop. 24.

Let it be required to find the attraction of the solid thus generated by the polygon, when the curve pr is a circle,* and the attracted point at the vertex p of the solid.

The attraction of a regular polygon was found in *Prop.* 5 and it will be adapted to our present purpose, by putting x^2 for a^2, and $2kx - x^2$ for b^2, where k is the radius of the circle pr: and we have, for the attraction of the solid,

$$A = 2n\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{x^2}(2kx - x^2)}\right) - (n-2)\,\pi x$$

$$\text{or } A = 2n\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)\,x - r^2x^2}\right) - (n-2)\,\pi x.$$

That part of A, under the sign of integration, equals

$$2nx \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)\,x - r^2x^2}\right) - 2n\int x\, \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)\,x - r^2x^2}\right),$$

* We may, not improperly, term this solid *a polygonal sphere.*

and, if we put $z\dot{z} = \dot{x}$, the last term becomes

$$-2n\int \dot{z^2} \text{ arc } \left(\text{tang.} = \frac{1}{rz}\sqrt{2k(1+r^2) - r^2z^2}\right)$$

$$= 2n\int \frac{rz^2\dot{z}}{\sqrt{2k(1+r^2) - r^2z^2}} = 2n\int \frac{z^2\dot{z}}{\sqrt{\frac{2k(1+r^2)}{r^2} - z^2}}; \text{ this fluent is}$$

$$2n\left\{\frac{k(1+r^2)}{r^2} \text{ arc } \left(\text{sine} = \frac{rz}{\sqrt{2k(1+r^2)}}\right) - \frac{1}{2}z\sqrt{\frac{2k(1+r^2)}{r^2} - z^2}\right\}$$

or by putting its value for z,

$$\frac{2nk(1+r^2)}{r^2} \text{ arc } \left(\text{sine} = \frac{r\sqrt{x}}{\sqrt{2k(1+r^2)}}\right) - n\sqrt{\frac{2k(1+r^2)}{r^2}x - x^2}.$$

Collecting all the parts of A, we have at length

$$A = 2nx \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right) - \frac{n}{r}\sqrt{2k(1+r^2)x - r^2x^2} + \frac{2nk(1+r^2)}{r^2} \text{ arc } \left(\text{sine} = \frac{r\sqrt{x}}{\sqrt{2k(1+r^2)}}\right) - (n-2)\pi x + \text{corr.}$$

But it is easy to see that each of the arcs in this expression is the complement of the other; put then $A = \text{arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right)$ and the expression becomes

$$A = \left\{\frac{nk}{r^2} + n(k-x)\right\}(\pi - 2A) + 2x\pi - nx\,.\,\text{tang. } A + \text{corr.}$$

When $x = o$, $A = \frac{\pi}{2}$, so that, if the fluent is to begin when $x = o$, no correction is necessary.

When $x = 2k$, $A = \text{arc } \left(\text{tang.} = \frac{1}{r}\right) = \frac{(n-2)\pi}{2n}$, and $\sqrt{2k(1+r^2)x - r^2x^2} = 2k$, $\pi - 2A = \pi - \frac{(n-2)\pi}{n} = \frac{2\pi}{n}$: whence we have for *the attraction of the whole solid*

$$A = \frac{1+r^2}{r^2} \times 2k\pi - \frac{2nk}{r} \dots\dots\dots\dots (\alpha).$$

This will appear to be an expression of great simplicity, if we reflect what very different solids it belongs to, from that whose section is a triangle, to the sphere whose section is a circle.

In the latter case n is infinitely great and r infinitely small; r is also the tangent of $\frac{\pi}{n} = \frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}$, if we reject the other terms of the expansion, on account of their smallness.

The expression for A may then, in this case, be put into the form

$$A = 2k\pi + \frac{2k\pi}{r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}\right\}} - \frac{2nk}{r};$$

now $r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}\right\} = (q \cdot p)\, r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi}{n} r^2\right\} = r\left(1 + \frac{r^2}{3}\right)\frac{\pi}{n}$, whence the second term of the last number is changed into

$$\frac{2kn}{r\left(1 + \frac{r^2}{3}\right)} = \frac{2kn\left(1 - \frac{r^2}{3}\right)}{r} = \frac{2kn}{r} - \frac{2}{3}knr = \frac{2nk}{r} - \frac{2}{3}k\pi, \text{ by}$$

putting π for nr: and, by substituting this value, we have at last $A = \frac{4}{3}k\pi$; which is the well known expression for the attraction of a sphere on a point at its surface.

If the generating polygon is a square instead of a circle, $r =$ tang. $45^\circ = 1$, and equation (α) gives $A = 4k(\pi - 2)$ $= 4k \times 1{,}14159$, &c. which exceeds the attraction of the sphere by about one-tenth, if pr is the same circle in both.

Cor. If we would know the radius (k') of a sphere, which shall attract, a point at its surface, as much as a polygonal sphere, of the length $2k$, does a point at its vertex, we have only to put

$$\frac{4}{3}\pi k' = \frac{1+r^2}{r^2}\, 2\pi k - \frac{2nk}{r}, \text{ whence}$$

$$k' = \frac{3}{2}k\left\{\frac{1+r^2}{r^2} - \frac{n}{\pi r}\right\}.$$

Prop. 25.

Let the directing curve pr be a parabola; the attracted point at the vertex p of the solid.

We must here make use of the formula in *Prop.* 5, as we did in the last example; but as the equation of a parabola is $y^2 = \alpha x$, this latter quantity must be put for b^2. Thus we get, for the attraction of the solid,

$A = 2n f\dot{x}$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right) - (n-2)\pi x.$

The part, having the sign of integration, may be put under the form $2nx$ arc. $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right)$ $2n f x \, \dot{\text{arc}}$ $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right)$; in the last term of which put $z^2 = x$, and it will become

$$-2n f z^2 \, \dot{\text{arc}} \left(\text{tang.} = \frac{1}{rz}\sqrt{z^2 + (1+r^2)\alpha}\right)$$

$$= 2n\alpha r \int \left\{ \frac{\dot{z}}{\sqrt{z^2 + (1+r^2)\alpha}} - \frac{\alpha \dot{z}}{(z^2+\alpha)\sqrt{z^2 + (1+r^2)\alpha}} \right\}$$

$$= 2n\alpha r \left\{ L\left(z + \sqrt{z^2 + (1+r^2)\alpha}\right) - \frac{1}{r} \text{arc} \left(\text{sine} = \frac{r}{\sqrt{1+r^2}} \times \frac{z}{\sqrt{z^2+\alpha}}\right) \right\}.$$

Collecting all the terms, we have at length

$A = 2nx$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right) - 2n\alpha$ arc $\left(\text{sine} = \frac{r}{\sqrt{1+r^2}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}\right) + 2n\alpha r \, L\left(\sqrt{x} + \sqrt{x + (1+r^2)\alpha}\right) - (n-2)\pi x +$ corr.

It is observable here, as in the last proposition, that each of the arcs in this expression is the complement of the other; put $A =$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right)$, and the attrac-

tion becomes $A = n(x + \alpha)\, 2A - \pi) + 2x\pi + 2n\alpha r\, L\, (\sqrt{x} + \sqrt{x + (1 + r^2)\alpha}) +$ corr.

When $x = o$, $A = \frac{\pi}{2}$; so that, if the fluent is to begin at that term, we have corr. $= - 2n\alpha r\, L \sqrt{(1 + r^2)\alpha}$.

If we would find, from this expression, the attraction of the limit of these solids (which is the parabolic conoid) we must observe, that the arc A may be put under the form $\frac{\pi}{2} -$ arc $\left(\text{sine} = \frac{r}{\sqrt{1+r^2}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}\right)$, whence, because r is infinitely small, $2A - \pi = - 2r \cdot \frac{\sqrt{x}}{\sqrt{x+\alpha}} = - 2\frac{\pi}{n} \cdot \frac{\sqrt{x}}{\sqrt{x+\alpha}}$, qu. prox.; substituting this value, and π for nr, and neglecting r^2, we get

$$A = 2\pi \left\{x - \sqrt{x^2 + \alpha x} + \alpha L\, (\sqrt{x} + \sqrt{x + \alpha})\right\} + \text{corr.}$$

for the action of a parabolic conoid on a point at its vertex.

Prop. 26.

Let the curve pr be a parabola *convex* to the axis pm, in which case $y = \frac{x^2}{\alpha}$; and we have, by proceeding as before,

$A = 2n\int \dot{x}$ arc $\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{(1+r^2)}{\alpha^2} x^2}\right) - (n-2)\pi x$; or

$A = 2nx$ arc $\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{\alpha^2} x^2}\right)\, 2n\int x\, \dot{\overline{\text{arc}}}\, \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{\alpha^2} x^2}\right) - (n-2)\pi x$; if we put $a^2 = \frac{\alpha^2}{1+r^2}$, the term, under the integral sign, becomes

$$- 2n\int x\, \dot{\overline{\text{arc}}}\, \left(\text{tang.} = \frac{1}{ar}\sqrt{a^2 + x^2}\right)$$

$$= - 2nar \int \left\{\frac{\dot{x}}{\sqrt{a^2+x^2}} - \frac{(1+r^2)a^2\dot{x}}{\left\{(1+r^2)a^2 + x^2\right\}\sqrt{a^2+x^2}}\right\}$$

$$= - 2nar \left\{ L \left(x + \sqrt{a^2 + x^2}\right) - \frac{\sqrt{1+r^2}}{r} L \left(\frac{x}{\sqrt{x^2+a^2}} + \sqrt{\frac{1}{r^2} + \frac{x^2}{x^2+a^2}} \right) \right\}; \text{ so that}$$

$$A = 2nx \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} x^2}\right) - (n-2)\pi x + \text{corr.}$$

$$- 2nr \frac{a}{\sqrt{1+r^2}} L \left(x + \sqrt{\frac{a^2}{1+r^2} + x^2}\right) + 2naL \left\{ \frac{x}{\sqrt{x^2+a^2}} + \sqrt{\frac{1}{r^2} + \frac{x^2}{x^2+a^2}} \right\}.$$

This is the attraction of a polygonal parabolic spike, on a particle at the point. When the polygon becomes a circle,

$$\text{Arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} x^2}\right) = \frac{\pi}{2} - \frac{ra}{\sqrt{a^2+x^2}}, \text{ and}$$

$$L \left\{ \frac{x}{\sqrt{x^2+a^2}} + \sqrt{\frac{1}{r^2} + \frac{x^2}{x^2+a^2}} \right\} = L\frac{1}{r} + \frac{rx}{\sqrt{a^2+x^2}}, \text{ whence it}$$

will easily appear that, in the case under consideration,

$$A = 2\pi \left\{ x - aL \left(x + \sqrt{a^2 + x^2}\right) \right\} + \text{corr.}$$

We may conceive the plane uvv′u′, fig. 15, instead of a regular polygon to be a rectangle, moving along the line pm, as in the former case, with its centre in that line; and, with the middle points r and r′ of its sides, touching curves pr, pr′ either of the same or different kinds.

The section of the generated solid, or *groin*, perpendicular to its axis, will have its action on the point p (if we put $x =$ pm, and b and b' for the sides of the rectangle) expressed by

$$4 \text{ arc } \left(\text{tang.} = \frac{bb'}{x\sqrt{x^2+b^2+b'^2}}\right);$$

and if we multiply this by $\dot{x}$, and put for b and b' their values, given by the equations of the curves pr, pr′, the fluent will be the attraction of the generated solid.

Prop. 27.

Let pr be a circle, pr′ a parabola; to find the attraction of the solid on a point p at its vertex.

Let the radius of the circle be k, the parameter of the parabola α: then we have $b = \sqrt{2kx - x^2}$; $b' = \sqrt{\alpha x}$, and

$$A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{\sqrt{\alpha x}\sqrt{2kx - x^2}}{x\sqrt{(2k+\alpha)x}}\right),$$

or $A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right)$, if we put $c = \frac{\sqrt{\alpha}}{\sqrt{2k+\alpha}}$, and taking the fluent by parts

$A = 4x \text{ arc } \left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right) - 4\int x \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right)$,

the last term of which, if we put z^2 for x, becomes

$-4\int z^2 \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{c\sqrt{2k-z^2}}{z}\right)$ and this, by actually taking the fluxion of the arc $= \int \frac{8kcz^2\dot{z}}{\left\{(1+c^2)z^2 + 2kc^2\right\}\sqrt{2k-z^2}}$, or,

by restoring the value of c, $= 4\sqrt{\alpha} \,.\, \sqrt{2k+\alpha} \,.\int \frac{z^2\dot{z}}{(z^2+\alpha)\sqrt{2k-z^2}}$,

or by division

$$= 4\sqrt{\alpha}\,.\,\sqrt{2k+\alpha}\,.\int\left\{\frac{\dot{z}}{\sqrt{2k-z^2}} - \frac{\alpha\dot{z}}{(z^2+\alpha)\sqrt{2k-z^2}}\right\}$$

$= 4\sqrt{\alpha}\sqrt{2k+\alpha} \text{ arc } \left(\text{sine} = \frac{z}{\sqrt{2k}}\right) - 4\alpha \text{ arc } \left(\text{sine} = \frac{\sqrt{2k+\alpha}}{\sqrt{2k}} \times \frac{z}{\sqrt{z^2+\alpha}}\right)$;

so that we have at last

$A = 4x \text{ arc } \left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{2k+\alpha}} \times \frac{\sqrt{2k-x}}{\sqrt{x}}\right) + 4\sqrt{\alpha}\sqrt{2k+\alpha} \text{ arc } \left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) - 4\alpha \text{ arc } \left(\text{sine} = \frac{\sqrt{2k+\alpha}}{\sqrt{2k}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}\right) + \text{corr.} \ldots\ldots (\beta).$

As this expression vanishes when $x = o$, if the fluent is to

begin at that value of x, no correction is to be added. The last of the arcs, in the expression above, is the complement of the first; denote then the first by A, and we have

$$A = 4(x + \alpha)A - 2\alpha\pi + 4\sqrt{\alpha}\sqrt{2k + \alpha}\,\text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right);$$

lastly, if we want the whole fluent, when $x = 2k$, we get

$$A = 2\pi(\sqrt{2k\alpha + \alpha^2} - \alpha).$$

Cor. 1. If we make α infinite, in this last value of A, it becomes $A = 2k\pi$; which is the action of *an infinitely long circular cylinder*, on a point at its surface. This is the attraction of the whole cylinder when $x = 2k$; to find the same for *any* value of x, make α infinite in formula (β); this gives

$$A = 4x\,\text{arc}\left(\text{tang.} = \frac{\sqrt{2k - x}}{\sqrt{x}}\right) + 4(\alpha + k)\,\text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right)$$

$$- 4\alpha\,\text{arc}\left\{\text{sine} = \left(1 + \frac{2k - x}{2\alpha}\right)\frac{\sqrt{x}}{\sqrt{2k}}\right\};$$ but (Euleri Calc. Diff. p. 376)

$$\text{Arc}\left\{\text{sine} = \left(1 + \frac{2k - x}{2\alpha}\right)\frac{\sqrt{x}}{\sqrt{2k}}\right\} = \text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) + \frac{\sqrt{2kx - x^2}}{2\alpha}$$

qu. prox.; the substitution of this value gives

$$A = 4x\,\text{arc}\left(\text{tang.} = \frac{\sqrt{2k - x}}{\sqrt{x}}\right) + 4k\,\text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) - 2\sqrt{2kx - x^2},$$

which, because the latter arc is the complement of the former, is changed to

$$A = 2k\pi - 4(k - x)\,\text{arc}\left(\text{tang.} = \frac{\sqrt{2k - x}}{\sqrt{x}}\right) - 2\sqrt{2kx - x^2}.$$

Cor. 2. In like manner we may find the *attraction of an infinitely long parabolic cylinder*, on a point in its surface, at the vertex of the parabola; this is effected by making k infinite in formula (β), whence there results

$$A = 4x\,\text{arc}\left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{x}}\right) - 4\alpha\,\text{arc}\left(\text{tang.} = \frac{\sqrt{x}}{\sqrt{\alpha}}\right) + 4\sqrt{\alpha x};$$ or

$$A = 4(x+\alpha) \text{ arc } \left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{x}}\right) + 4\sqrt{\alpha x} - 2\alpha\pi.$$

Cor. 3. If $\alpha = \frac{2}{3}k$, the attraction of the solid in the proposition equals that of a sphere whose radius is k; for by substituting $\frac{2}{3}k$ for α, in the expression $2\pi(\sqrt{2k\alpha + \alpha^2} - \alpha)$, it becomes $\frac{4}{3}k\pi$, the action of a sphere whose radius is k on a point in its surface.

The attractions of cylinders of finite length, in directions perpendicular to their axes, are to be found after the manner of this last proposition; but there are not many cases in which they can be expressed by circular arcs and logarithms.

Prop. 28.

Let fig. 16 represent a circle, C the centre, ab, cd two parallel chords; conceive a right cylinder, whose section is the portion abcd of the circle, terminated by the chords ab, cd, to be extended to the distance d above and below the plane of the figure.

It is required to determine the action of this cylinder on a point at C.

Put k for the radius of the circle, and let x be the distance from C of a chord parallel to ab. Then, using the same formula as in the last *Prop.* we have $b = d$, $b' = \sqrt{k^2 - x^2}$, and for the action of the solid, $A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{d\sqrt{k^2-x^2}}{x\sqrt{d^2+k^2}}\right)$ or

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{d\sqrt{k^2-x^2}}{x\sqrt{d^2+k^2}}\right) - 4\int x \, \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{d\sqrt{k^2-x^2}}{x\sqrt{d^2+k^2}}\right)$$

$$= 4x \text{ arc } \left(\text{tang.} = \frac{d\sqrt{k^2-x^2}}{x\sqrt{d^2+k^2}}\right) - 4d\text{L}.\frac{\sqrt{d^2+k^2} + \sqrt{k^2-x^2}}{\sqrt{d^2+x^2}} + \text{corr.}$$

If the fluent is to begin when $x = o$, the correction is

$4dL \cdot \frac{\sqrt{d^2+k^2}+k}{d}$; and the fluent, taken from $x=o$ to $x=k$, is

$$A = 4dL \frac{\sqrt{d^2+k^2}+k}{d},$$

which agrees with what Mr. PLAYFAIR found in a different manner; except in this case, the object of the present proposition is different from that of his, which finds the action of such portions of the cylinder as have *sectors* for their bases.

Prop. 29.

Let the base of the cylinder be the figure parmnbp, fig. 17, the curves par, pbn being inverted parabolas; or in which $\text{pm}^2 = \alpha \times \text{rm}$. Let the attracted point be at p; and let the cylinder be extended to the distance d above and below the plane of the figure.

Using the same formula as before, and putting $\text{pm} = x$, we have $b = d$, $b' = \frac{x^2}{\alpha}$; and, for the action of the solid,

$$A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{dx}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right), \text{ or}$$

$A = 4x$ arc $\left(\text{tang.} = \frac{ux}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right) - 4\int x \, \overline{\text{arc}} \left(\text{tang.} = \frac{dx}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right)$;

the last term of which becomes, by taking the fluxion of the arc,

$$-4\int \frac{\alpha^2 d^3 x\dot{x} - dx^5\dot{x}}{\left\{\alpha^2 d^2 + (\alpha^2 + d^2)x^2 + x^4\right\} \sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}$$

an expression integrable by circular arcs and logarithms.

When d is infinite, this fluent takes a very simple form, viz.

$$-4\int \frac{\alpha x\dot{x}}{\alpha^2 + x^2} = -2\alpha \int \frac{\frac{2x\dot{x}}{\alpha^2}}{1 + \frac{x^2}{\alpha^2}} = -2\alpha L \left(1 + \frac{x^2}{\alpha^2}\right), \text{ and, in}$$

this case

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{x}{\alpha}\right) - 2\alpha L \left(1 + \frac{x^2}{\alpha^2}\right).$$

Cor. 1. Draw the right lines pr, pn; because $\frac{x}{\alpha} = \frac{\text{pm}}{\alpha} = \frac{\text{rm}}{\text{pm}}$ = tang, rpm, the last expression may be put into the form

$$A = 4x \text{ arc. rpm} - 4\alpha L \text{. sec. rpm.}$$

Cor. 2. If, in the last value of B in *Prop.* 20, we make x infinite, there results $B = x'$ arc (tang. $= r$); from whence it is plain that the first term of the expression for A in the last cor. viz. $4x$ arc rpm, expresses the action of an infinitely long prism, whose base is the triangle rpn, on the point p.

Consequently, the other term of A, or $4\alpha L$. sect. rpm, is the action of the infinitely long solid whose base consists of the parabolic segments parp, pbnp.

We may next consider the generating plane uvv′u′, fig. 15, to be a rhombus, given in *species*, and so varying in magnitude, as to touch four similar and equal curves, at those points where perpendiculars from the centre of the rhombus fall on its sides.

Prop. 30.

Let the guiding curves be semi-circles, to the radius k; the attracted point at the vertex p.

We saw, in *Prop.* 2, *Cor.* 1, that the action of a rhombus, on a point placed perpendicularly over its centre, is $A = 4$ arc $\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) + 4 \text{ arc } \left(\text{tang.} = \frac{1}{r'}\sqrt{1 + \frac{1+r'^2}{a^2} b^2}\right) - 2\pi$; in which we must put x^2 for a^2, $2kx - x^2$ for b^2, and we get, for the attraction of the solid,

$$A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right) + 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r'x}\sqrt{2k(1+r'^2)x - r'^2x^2}\right) - 2\pi x.$$

These fluents being exactly similar to the one in *Prop.* 24, if we put $A = \text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right)$

$$A' = \text{arc}\left(\text{tang.} = \frac{1}{r'x}\sqrt{2k(1+r'^2)x - r'^2x^2}\right)$$

it is easy to see that

$$A = \left\{\frac{2k}{r^2} + 2(k-x)\right\}(\pi - 2A) - \frac{2}{r}\sqrt{2k(1+r^2)x - r^2x^2}$$
$$+ 2\pi x$$
$$+ \left\{\frac{2k}{r'^2} + 2(k-x)\right\}(\pi - 2A') - \frac{2}{r'}\sqrt{2k(1+r'^2)x - r'^2x^2}$$
$$+ \text{corr.}$$

If the fluent is to begin when $x = o$, no correction is necessary; for at that term $A = A' = \frac{\pi}{2}$.

When $x = 2k$, $A = \text{arc}\left(\text{tang.} = \frac{1}{r}\right)$, $A' = \text{arc}\left(\text{tang.} = \frac{1}{r'}\right)$ and

$$A = \left\{\frac{2k(1-r^2)}{r^2}\right\}\left\{\pi - 2\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right)\right\} - \frac{4k}{r} + 4k\pi$$
$$+ \left\{\frac{2k(1-r'^2)}{r'^2}\right\}\left\{\pi - 2\,\text{arc}\left(\text{tang.} = \frac{1}{r'}\right)\right\} - \frac{4k}{r'}.$$

If we thought proper, this might still be put under a different form; for $r' = \frac{1}{r}$, and the arcs the complements of each other; and $\pi - 2\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right) = 2\,\text{arc}\,(\text{tang.} = r)$; also $\pi - 2\,\text{arc}\left(\text{tang.} = \frac{1}{r'}\right) = 2\,\text{arc}\,(\text{tang.} = r')$.

Cor. 1. When the rhombus is a square, $r = r' = 1$; and the action becomes $A = 4k\pi - 8k$, as we found in *Prop.* 24.

Cor. 2. Let r' be infinite, then $r = o$, and the solid becomes an infinitely long circular cylinder; and it is easy to see that the value of A is reduced to $2k\pi$, as we found before in a different manner.

The foregoing problems, which I have chosen from a great

variety that occurred to me, are sufficient to shew the use that may be made of the expressions given in the first section.

The attractions of certain infinitely long cylinders, which were derived, as corollaries, from some of the preceding propositions, present us with several curious relations; with these I shall terminate the present division of my subject.

Let pout, fig. 18, represent the base, or section, of a circular cylinder, infinitely extended both above and below the plane of the figure. Let p be an attracted point in the circumference of the section. Draw the diameter pu, and, at right angles to it, the diameter ot.

By *Cor.* 1, *Prop.* 27, the action of the whole cylinder on the point p is $2k\pi$ (k being the radius of the circular section); the action of that half of the cylinder, whose base is the semicircle opto, is $2k\ (\pi - 1)$; the action of the other half of the cylinder, which is furthest from p, is $2k$: therefore,

1. The attraction of a sphere is to that of an infinite circular cylinder of the same diameter (on a point at the surface of each) as $\frac{2}{3}$ to 1, which is the ratio of the solidity of a sphere to that of its circumscribing cylinder.

2. The attraction of the whole infinite cylinder, on p, is to the attraction of that half which is furthest from that point, as the circumference of a circle is to its diameter.

3. Consequently, the attraction of the nearest half, is to that of the furthest half, as the difference between the circumference and diameter of a circle is to the diameter; or *nearly* as 2 to 1.

4. In the circle optu, fig. 18, inscribe the parabola owpvt, whose equation is $kx = y^2$, so that its vertex may be at p, and its axis coincide with pu: this parabola will plainly cut the

circle at the quadrantal points o and t; and, I say, that the action, on the point p, of the infinitely long cylinder, whose base is the parabolic area owpvtco, is to the attraction of the furthest half of the infinite circular cylinder, *exactly* as 2 to 1. For the latter action has been shewn to be $2k$; and if in the expression, obtained in *Cor.* 2, *Prop.* 27, we make α and x both $= k$, it is reduced to $A = 4k$.

5. In fig. 18, draw fug perpendicular to pu at u; and from p, through o and t, the lines pof, ptg. The attraction, on the point p, of the infinitely long prism whose base is the triangle pfg, is equal to the attraction of the infinitely long circular cylinder. For the action of the prism is $4 \times$ pu $\times$ arc. fpu (by *Prop.* 20) $= 8k \times \frac{\pi}{4} = 2k\pi$; and this has been already shewn to be the attraction of the circular solid.

§. V.

Of Solids of greatest Attraction.

The subject of this section has occupied the attention of Mr. Playfair,* in the same paper I have before noticed; it had previously been treated of by Silvabelle. Frisi also, in the third volume of his works, gives a solution of the same problem as that which is first considered by Mr. Playfair, but his result is an erroneous one. None of these writers have pursued the matter any further than what relates to the figure of a homogeneous solid of revolution. My manner of treat-

* The problems which I investigate are similar to *the first* of Mr. Playfair's, where the *equation of a curve* is sought; nor do I at all meddle with that other class of problems which he treats of in the subsequent part of the paper.

ing the subject connects it intimately with the preceding parts of this paper; otherwise I should not have given the following problems a place here.

Prop. 31.

Suppose that a given quantity of matter is to be formed into a right cylinder of the length $2d$; what must be the figure of its base, so that it shall attract, with the greatest force possible, a point in its surface, and in the middle with respect to its two ends?

Let fig. 19 represent a section of the cylinder, at the attracted point p, parallel to its base. It is plain enough, that, whatever is the nature of the curve pab, we may draw a line pb from p, which shall divide the area into two equal and similar portions pabp, pcbp.

Put the absciss pd $= x$, the ordinate ad $= y$: the mass of the cylinder is $4d\int y\dot{x}$; and, by *Prop.* 4, its attraction on p is

$$4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{dy}{x\sqrt{d^2+x^2+y^2}}\right).$$

Let C be a constant quantity, and we have only to make the fluxion of the following expression, with respect to y, equal to nothing,* viz.

$$\text{Arc } \left(\text{tang.} = \frac{dy}{x\sqrt{d^2+x^2+y^2}}\right) + \text{C}dy:$$

this gives

$$\frac{x}{(x^2+y^2)\sqrt{d^2+x^2+y^2}} + \text{C} = o, \text{ or } x^2 - \text{C}^2 (x^2+y^2)^2 (d^2+x^2+y^2) = o,$$

for the equation of the curve pab. Make $y = o$, and let a be the corresponding value of x; the equation becomes

$1 - \text{C}^2 a^2 (d^2 + a^2) = o$, whence $\text{C}^2 = \frac{1}{a^2(d^2+a^2)}$; by substituting

* EULERI "Methodus, &c." p. 42 and 185-6-7.

which value in the equation of the curve, it is ultimately

$$a^2 (d^2 + a^2) x^2 - (x^2 + y^2)^2 (d^2 + x^2 + y^2) = o.$$

It is proper to remark, that though, in the enunciation, I spoke of the point as being in the surface of the cylinder, yet there is nothing in the above method of investigation that supposes it to be in contact with the solid: if there is to be a given distance between them, the nature of the curve will be the same.

Cor. 1. If we make d infinitely small, there results $a^4x^2 - (x^2 + y^2)^3 = o$ for the equation of the curve bounding *the plane of greatest attraction;* and it is evident, that, by the revolution of this curve about its axis, will be generated *the solid of greatest attraction*, when it is sought for without any such conditions or restrictions as entered into the preceding problem.

This exactly agrees with the conclusion arrived at by SILVABELLE and Mr. PLAYFAIR.

Cor. 2. If, on the other hand, we make d infinitely great, the equation is reduced to $y^2 = ax - x^2$, which is that of a circle whose diameter is a, the attracted point being in the circumference. Therefore,—*of all infinitely long cylinders, having the areas of their bases, or transverse sections, equal, that which has a circle for the circumference of the said base, shall exert the greatest action on a point at its surface.*

Prop. 32.

Let a given quantity of matter be fashioned into such a solid as was treated of at the beginning of the last section (in Propositions 24, 25, 26), viz. having its section perpendicular to the axis a regular polygon. The polygon being given in

species, it is required to determine the nature of the curve pr, fig. 15, so that the solid may have the greatest possible action on a point at its vertex p.

If we put x for the distance of the generating polygon from the vertex, and y for the perpendicular let fall from the centre of the polygon on one of its sides, the action of the solid is

$$2n\int\dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{x^2}y^2}\right) - \int(n-2)\pi\dot{x}$$

by *Prop.* 5, and the mass of the solid is $nr\int y^2\dot{x}$; so that the quantity whose fluxion, with respect to y, must $= 0$, is

$$2n \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{x^2}y^2}\right) + Cnry^2, \text{ and we get}$$

$$\frac{x}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + C = 0, \text{ or } x^2 - C^2(x^2+y^2)^2(x^2+(1+r^2)y^2) = 0.$$

Let a be the value of x when $y = 0$, then $C^2 = \frac{1}{a^4}$, and the equation of the curve becomes

$$a^4x^2 - (x^2+y^2)^2\left\{x^2+(1+r^2)y^2\right\} = 0.$$

When the polygon is a circle, $r = 0$, and the equation is reduced to $a^4x^2 - (x^2+y^2)^3 = 0$, the same as we found in *Cor.* 1, *Prop.* 31.

Lemma 1.

To find the attraction of the right prism whose base is the triangle mrv, fig. 1, and height $\dot{a}$, on the point p, in the direction pm; on the supposition, that the density at the ordinate ks is as any function of the absciss mk, and distance pm.

If we use the same notation as in *Prop.* 1, and put f (a, T) for the density of a particle any where at the line ks, we shall find, by proceeding, as we did there,

$$A = \dot{a}\int\frac{arf(a,T)\,T\dot{T}}{(a^2+T^2)(a^2+(1+r^2)T^2)^{\frac{1}{2}}}.$$

Hence the attraction of a prism, whose height is $\dot{a}$, and base a regular polygon of n sides, composed of triangles having such a law of density as was supposed above, will be, on a point placed perpendicularly over its centre,

$$A = 2n\dot{a}\int \frac{arf(a,T)\,T\dot{T}}{(a^2+T^2)\,(a^2+(1+r^2)T^2)^{\frac{1}{2}}}.$$

This expression would be easily integrable on various suppositions. Thus we might conceive the density at the ordinate ks to vary as the line ps, drawn from the attracted point to its extremity s; this would be to make f $(a,T) = \sqrt{a^2+(1+r^2)T^2}$; whence $A = 2n\dot{a}\int \frac{arT\dot{T}}{a^2+T^2} = rna\dot{a}\left\{L.(a^2+T^2) - L.a^2\right\}$.

Again, we might suppose f $(a,T) = a^2 + T^2 = (\text{pk})^2$; this would give $A = 2n\dot{a}\int \frac{arT\dot{T}}{\sqrt{a^2+(1+r^2)T^2}} = \frac{2nra\dot{a}}{1+r^2}\left\{\sqrt{a^2+(1+r^2)T^2} - a\right\}$

But the kind of problems we are engaged about does not require us to know the value of A itself; its fluxional coefficient with respect to T being alone wanted, and this is *always*

$$\frac{2narf(a,T)\,T}{(a^2+T^2)\sqrt{a^2+(1+r^2)T^2}} \times \dot{a} \text{ .... } (1).$$

For suppose we had actually found the fluent; when we make use of it in such a problem as the last, we must change T into y, and take the fluxion with respect to y, and the result must necessarily be $\frac{2narf(a,y)\,y}{(a^2+y^2)\sqrt{a^2+(1+r^2)y^2}} \times \dot{a}$; which we might have arrived at simply by changing T into y in the expression marked (1).

Lemma 2.

To find the quantity of matter in a right prism, whose base is the triangle rmv, fig. 1, and height $\dot{a}$; supposing the density at any ordinate ks to be f (a,T).

The *magnitude* of the element of the prism is $\dot{a}Tr\dot{T}$, and the *mass* of this element is $r\dot{a} \times \mathrm{f}\,(a,T)\,T\dot{T}$; whence the mass of the whole prism is $r\dot{a}\int \mathrm{f}\,(a,T)\,T\dot{T}$.

The mass of a prism whose height is $\dot{a}$ and base a regular polygon of n sides, formed of triangles having this law of density, is $2nr\dot{a}\int \mathrm{f}\,(a,T)\,T\dot{T}$ and its fluxional coefficient, with respect to T, is $2nr\,\mathrm{f}\,(a,T)\,T \times \dot{a}$.

Prop. 33.

Let the last proposition be again proposed, but with this difference, that the solid, instead of being homogeneous, is to be formed of polygonal prismatic elements, having such a law of density as in the preceding lemmas.

By proceeding as before, we shall have for the equation of the curve pr, fig. 15,

$$\frac{2nxr\,\mathrm{f}\,(x,y)\,y}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + C2nr\,\mathrm{f}\,(x,y)\,y = o,$$

or $\frac{x}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + C = o$; *which is exactly the same equation as when the solid was supposed to be homogeneous.*

When $r = o$, we have, as before, $a^4x^2 = (x^2 + y^2)^2$; which shews that *the result of Mr.* PLAYFAIR *extends to an infinity of cases besides that of homogeneity.*

When, as in our last supposition, $r = o$, and the mass is a solid of revolution, the function $\mathrm{f}\,(a,T)$ expressing the density, is a function of the perpendicular let fall from any particle on the axis of the solid, and of the distance between the foot of that perpendicular and the attracted point.

Scholium.

If the preceding lemmas had been treated on the supposition that the density was variable *along* the line ks, fig. 1, (which is the same as making the density f (t), or more generally f (a, T, t) a function of a, T, and t) their application to the problem we have been considering, would give an indefinite number of different equations, for the curve pr, fig. 15, according to the nature of the assumed function f (a, T, t): every one of which equations will, however, have this peculiarity, that if we make $r = o$, it will become $a^4x^2 = (x^2 + y^2)^3$. For when $r = o$, $t = a$, and f (a, T, t) becomes a function of a and T only, and the case enters into *Prop.* 33 just now considered.

It may be worth while to see an example of this; we should have had, in general, for the action of the polygonal prismatic element of the solid, by *Prop.* 1,

$$A = 2n\dot{a}\int\int\frac{a\,\mathrm{f}\,(a, T, t)\,\dot{T}\ddot{t}}{(a^2+T^2+t^2)^{\frac{3}{2}}};$$

and the mass of the same element would have been

$$2n\dot{a}\int\int \mathrm{f}\,(a, T, t)\,\dot{T}\dot{t}.$$

These must be integrated, with respect to t, from $t = o$ to $t = rT$: which cannot be done till we assign a form for the function f (a, T, t). Let this be $a^2 + T^2 + t^2$, that is to say, let the density at any point q, in the triangle vrm, be as the square of its distance pq from the attracted point p. This will give

$$A = 2n\dot{a}\int\int\frac{a\dot{T}\ddot{t}}{(a^2+T^2+t^2)^{\frac{1}{2}}} = 2na\dot{a}\int\left\{\mathrm{L}\,(rT + \sqrt{a^2 + (1+r^2)\,T^2}) - \mathrm{L}\sqrt{a^2+T^2}\right\}\dot{T};$$ and for the mass

$$2n\dot{a}ff\,(a^2 + T^2 + t^2)\,\dot{T}i = 2n\dot{a}f\left(a^2rT + rT^3 + \frac{r^3T^3}{3}\right)\dot{T}.$$

If therefore we solve *Prop.* 32, on this supposition of density, we have for the equation of the curve pr, fig. 15, when the solid has the greatest attraction,

$$xL\left(ry\sqrt{x^2 + (1 + r^2)\,y^2}\right) - xL\sqrt{x^2 + y^2} + C\left(x^2ry + ry^3 + \frac{r^3y^3}{3}\right) = 0.$$

Now, when r is infinitely small, we shall have, by neglecting all the higher powers thereof,

$$L\left(ry + \sqrt{x^2 + (1 + r^2)y^2}\right) = L\sqrt{x^2 + y^2} + \frac{ry}{\sqrt{x^2+y^2}};$$

by substituting which our equation becomes

$\frac{x}{\sqrt{x^2+y^2}} + C\,(x^2 + y^2)$, or $a^4x^2 - (x^2 + y^2)^3 = 0$, as we shewed *a priori* must necessarily happen.

I shall just remark here, that, as the results of *Prop.* 32, are not altered by conceiving the density any function of a and T, such is also the case with respect to Problem 31, if T there represent the distance of any particle from a plane passing through the attracted point and the axis of the cylinder. This the reader may easily convince himself of.

The proposition just mentioned (31) is only a particular case of the following very general one.

Prop. 34.

Let uvv′u′, fig. 15, be a rectangle, whose plane is perpendicular to the line pm, and its centre in that line. Let this rectangle move parallel to itself, in the direction pm, and vary in such a manner, that the middle points r and r′ of its sides may continually touch two different curves.

The quantity of matter in the solid so generated being given, and the nature of one of the curves as pr′, to find what must be the other curve pr, so that the action of the solid, on a point at its vertex p, may be the greatest possible.

Put x for the absciss pm, y' and y for the ordinates mr′, mr; then by *Prop.* 4, the action of the solid will be

$$4\int\dot{x}\ \text{arc}\left(\text{tang.} = \frac{yy'}{x\sqrt{x^2+y^2+y'^2}}\right);\ \text{and its mass is}\ 4\int yy'\dot{x}.$$

But y' is a given function of x, suppose $\mathrm{f}(x)$. The quantity

$$\text{Arc}\left(\text{tang.} = \frac{y\mathrm{f}(x)}{x\sqrt{x^2+y^2+\mathrm{f}(x)^2}}\right) + Cy\mathrm{f}(x)$$

is therefore to have its fluxion, with respect to y, made $= o$: and this gives, for the equation of the curve pr,

$$\frac{x}{(x^2+y^2)\sqrt{x^2+y^2+\mathrm{f}(x)^2}} + C = o,\ \text{or}\ x^2 - C^2(x^2+y^2)^2(x^2+y^2+\mathrm{f}(x)^2) = o.$$

Ex. 1. Let $\mathrm{f}(x) = ax$, or pr′ be a straight line, the equation of pr must be $x^2 - C^2(x^2+y^2)^2\left\{(1+a^2)x^2+y^2\right\} = o.$

Ex. 2. Let pr′ be a circle, or $\mathrm{f}(x)^2 = 2kx - x^2$, k being the radius, then $x^2 - C^2(x^2+y^2)^2(2kx+y^2) = o$, is the equation of the other curve.

Ex. 3. If pr′ is a parabola, or $\mathrm{f}(x)^2 = ax$, the equation of pr is $x^2 - C^2(x^2+y^2)^2(ax+x^2+y^2) = o.$

Scholium.

In *Prop.* 27, after having found the action of the solid there treated of, we derived, as corollaries, the action of parabolic and circular cylinders of infinite length, by separately making infinite the diameter of the circle and the parameter of the parabola. Perhaps it might therefore be supposed, that if we made k infinite in the second of the preceding examples, or a

infinite in the third, the result would be the equation of the base of the infinitely long *cylinder* of greatest attraction; which however is by no means the case; for that was found to be a circle, whereas the equation we get here is

$$x - C'(x^2 + y^2) = o,$$

and if we make a infinitely great in the first example, the equation becomes $C' = x^2 + y^2$, or the line pr is a circle with its centre at the attracted point.

We might resolve this problem, on a variety of hypotheses respecting the density; or we might add other conditions of a different kind; for instance, not only the mass of the solid, but the area of the section, passing through the required curve pr and axis pm, might be supposed constant. But I pass on to other suppositions respecting the force of attraction; which will be treated with as much brevity as possible.

Lemma 3.

To find the attraction of the triangle vrm, fig. 1, on the point p, in the direction pm, supposing the force to be inversely as the mth power of the distance.

Keeping the same notation as in *Prop.* 1, we have, for the attraction of an element at q, $\frac{\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{m}{2}}}$; which being resolved, gives, for the force of the whole triangle, in the direction pm, $A = \iint \frac{a\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{m+1}{2}}}$; the fluent is to be taken from $t = o$ to $t = rT$, and we have

$$A = \int \frac{arT\dot{T}}{(a^2+T^2)(a^2+(1+r^2)T^2)^{\frac{m-1}{2}}} \Big\{ 1 - (2-m)\frac{r^2T^2}{3(a^2+T^2)} + (2-m)$$

$$(4-m)\frac{r^4T^4}{3.5(a^2+T^2)^2} - \&c. \Big\}.$$

The further integration, with respect to T, is not necessary for our purpose.

Cor. 1. If we multiply this by $2n$, it will be the attraction of a regular polygon of n sides; and making n infinitely great and r infinitely small, the attraction of a circle to the radius T is found to be

$$A = \int \frac{2nraT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}} = \frac{-2nra}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}} + \frac{2nra}{(m-1)a^{m-1}}$$

which, by putting π for nr, is

$$A = 2\pi \left\{ \frac{1}{(m-1)a^{m-2}} - \frac{a}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}} \right\}$$

the same as is found differently by other writers.

Cor. 2. When r becomes infinite, the triangle vrm is changed into a parallelogram, infinitely extended in the direction rv; and we have

$$A = \int \frac{a\dot{T}}{(a^2+T^2)(rT)^{m-2}} \left\{ 1 - (2-m)\frac{r^2T^2}{3(a^2+T^2)} + (2-m)(4-m) \frac{r^4T^4}{3\cdot5\,(a^2+T^2)^2} - \&c. \right\}$$

which, when m is an even positive whole number greater than 2, is reduced to $A = \frac{2.4.6......(m-2)}{3.5.7......(m-1)} \int \frac{a\dot{T}}{(a^2+T^2)^{\frac{m}{2}}}$.

Cor. 3. If instead of the action of the triangle vrm, that of a rectangle, whose sides are rm (y) and rv (y'), had been required, we must have proceeded exactly in the same manner, but the fluent, with respect to t, must have been taken from $t = o$, to $t = y'$; so that we have only to substitute y' for rT, in the value found by the lemma, and there results

$$A = \int \frac{ay\dot{T}}{(a^2+T^2)(a^2+T^2+y'^2)^{\frac{m-1}{2}}} \left\{ 1 - (2-m)\frac{y'^2}{3(a^2+T^2)} + (2-m)(4-m)\frac{y'^4}{3\cdot5\,(a^2+T^2)^2} - \&c. \right\}$$

Another Method of finding the Action of the Triangle vrm.

The expressions we have found will terminate only when m is one of the series of numbers 2, 4, 6, &c. If m is among the odd numbers 1, 3, 5, &c. $\frac{m+1}{2}$ will be a whole positive number; and we have for the action of the triangle, or rectangle (accordingly as the fluent, with respect to t, is taken from $t = o$, to $t = rT$, or from $t = o$, to $t = y'$) provided m is greater than 3,

$$A = \int a\dot{T}\left\{\frac{1}{(m-1)(a^2+T^2)} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-1}{2}}} + \frac{m-2}{(m-1)(m-3)(a^2+T^2)^2} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-3}{2}}} + \frac{(m-2)(m-4)}{(m-1)(m-3)(m-5)(a^2+T^2)^3} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-5}{2}}} + \ldots\ldots + \frac{(m-2)(m-4)\ldots\ldots 3}{(m-1)(m-3)(m-5)\ldots\ldots 2\,(a^2+T^2)^{\frac{m-1}{2}}} \times \frac{t}{a^2+T^2+t^2} + \frac{(m-2)(m-4)\ldots\ldots 3}{(m-1)(m-3)(m-5)\ldots\ldots 2\,(a^2+T^2)^{\frac{m-1}{2}}} \times \frac{1}{\sqrt{a^2+T^2}} \times \text{arc}\left(\text{tang.} = \frac{t}{\sqrt{a^2+T^2}}\right)\right\}.$$

Let us, for brevity, denote this quantity by $\int a\dot{T}\phi\,(a, m, T, t)$; then for the action of the triangle vrm we have $A = \int a\dot{T}\phi\,(a, m, T, rT)$; and for the rectangle, whose sides are rm (y) and rv (y') $A = \int a\dot{T}\phi\,(a, m, T, y')$. When $m = 1$, or 3, the above expression will not give the attraction; but we evidently have, in the case of $m = 1$,

$$A = \int \frac{a\dot{T}}{\sqrt{a^2+T^2}} \times \text{arc}\left(\text{tang.} = \frac{t}{\sqrt{a^2+T^2}}\right);$$ and when $m = 3$,

$$A = \int a\dot{T}\left\{\frac{1}{2(a^2+T^2)} \times \frac{t}{a^2+T^2+t^2} + \frac{1}{2(a^2+T^2)^{\frac{3}{2}}} \times \text{arc}\left(\text{tang.} = \frac{t}{\sqrt{a^2+T^2}}\right)\right\}.$$

Cor. 1. For a polygon of n sides, these expressions must be multiplied by $2n$ as usual; and when m is greater than 3, $A = 2nfa\dot{T}\phi\ (a, m, T, rT)$, if in this we make n infinitely great and r infinitely small, it ought to enter into the general case of the attraction of a circle given in *Cor.* 1, to the first part of the lemma: and in fact we get

$$A = \left\{\frac{1}{m-1} + \frac{m-2}{(m-1)(m-3)} + \frac{(m-2)(m-4)}{(m-1)(m-3)\,m-5} + \cdots\cdots + \frac{(m-2)(m-4)\ldots 3}{(m-1)(m-3)\ldots 2} + \frac{(m-2)(m-4)\ldots 3}{(m-1)(m-3)\ldots 2}\right\} \times \int \frac{2narT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}},$$

or, because the quantity between the brackets is plainly equal to unity, becomes $A = \int \frac{2nraT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}}$ which is the same form as was found before for the general case.

Cor. 2. When r becomes infinite, and the triangle rmv is changed into an infinitely extended rectangle, we have for its attraction

$$A = \frac{3.5.7\ldots\ldots(m-2)}{2.4.6\ldots\ldots(m-1)} \int \frac{a\pi\dot{T}}{2\,(a^2+T^2)^{\frac{m}{2}}},$$

except when $m = 1$, in which case, $A = \int \frac{a\pi\dot{T}}{2\sqrt{a^2+T^2}}$.

Scholium.

This lemma has been treated on the supposition that the density is the same at every part of the triangle rmv, fig. 1: but there are other hypotheses which render the solution easier: for instance, we may conceive the density of a particle at q to be as its distance (t) from the line rm, in which case

$$A = \iint \frac{a\dot{T}\dot{t}t}{(a^2+T^2+t^2)^{\frac{m+1}{2}}} = \int \left\{\frac{-a\dot{T}}{(m-1)(a^2+T^2+t^2)^{\frac{m-1}{2}}} + \frac{a\dot{T}}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}}\right\}$$

where t must be made rT or y' accordingly as the action of a triangle or rectangle is required.

From this simple case, we may not only arrive at some curious results, connected with the particular hypothesis of density, but may find with equal ease the figure of a *homogeneous* solid of revolution of greatest attraction, as will just now be seen.

If the density was to be a function of a and T only, it would be sufficient to multiply the values found in the lemma, by that function, see lemma 1.

Prop. 35.

Let *Prop.* 31 be again proposed, but with this difference, that the force is inversely as the mth power of the distance, and that the density of any particle of the cylinder is as its distance (t) from that middle section (parallel to the ends of the cylinder) which passes through the attracted point.

In the expression we just now found, in the preceding scholium, put x for a, and d (half the length of the cylinder) for t. The action of the cylinder is

$$A = 4\iint \left\{ \frac{-x\dot{x}\dot{T}}{(m-1)(x^2+T^2+d^2)^{\frac{m-1}{2}}} + \frac{x\dot{x}\dot{T}}{(m-1)(x^2+T^2)^{\frac{m-1}{2}}} \right\};$$

its quantity of matter is $4\iiint \dot{x}\dot{T}\dot{t}t = 2\iint \dot{x}\dot{T}t^2$; so that we have, for the equation of the curve bounding the base,

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+d^2)^{\frac{m-1}{2}}} + Cd^2 = o.$$

Cor. 1. When d is infinitely small, this becomes

$$\frac{m-1}{2} \times \frac{xd^2}{(x^2+y^2)^{\frac{m+1}{2}}} + Cd^2, \text{ or } \frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} + C' = o.$$

Let a be the value of x when $y = o$, then $C' = -\frac{1}{a^m}$; and

the equation of the curve, bounding *the plane of greatest attraction*, is

$$a^m x = (x^2 + y^2)^{\frac{m+1}{2}},$$

which is exactly the same result as that obtained by Mr. PLAYFAIR, p. 203, on the supposition of *homogeneity*; and this was to be expected; for, though a certain condition of the density of the cylinder entered into the foregoing problem, yet when d vanishes, and the solid becomes a plane, we must evidently obtain the same result as if it had been arrived at by supposing the cylinder homogeneous; which in fact it will be when the length is evanescent.

Nor is this observation to be confined to that particular case when the density is as t: if we had solved the problem on the supposition of any function of x, T, and t, for the density, it is easy to see that though different functions will give different results when d is finite, yet when the solid becomes *a plane*, and $d = o$, the equation will always be reduced to

$$a^m x = (x^2 + y^2)^{\frac{m+1}{2}}.$$

Hence we may conclude, that, *the solid of revolution which shall exercise the greatest attraction on a point in its axis, when the force is inversely as the mth power of the distance, and the density either uniform, or any function whatever of x and T (T being the perpendicular let fall from any particle to the axis of the solid, and x the distance between the foot of that perpendicular and the attracted point) will have, for the equation of its generating curve,* $a^m x = (x^2 + y^2)^{\frac{m+1}{2}}$.

Cor. 2. Nothing can be learned from the equation

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+d^2)^{\frac{m-1}{2}}} + Cd^2 = o,$$

when $m = 1$. The curve is then transcendent, and has for its equation $x\mathrm{L}.(x^2 + y^2 + d^2) - x\mathrm{L}.(x^2 + y^2) + Cd^2 = o$.

Cor. 3. If the cylinder becomes infinitely long, (m being positive and greater than unity) the equation of its base is

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} + C' = o;$$

let a be the value of x when $y = o$; then $C' = \frac{-1}{a^{m-2}}$, and the equation becomes

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{1}{a^{m-2}} = o.$$

If $m = 2$, as in the case of nature, this becomes $\frac{x^2}{x^2+y^2} = 1$, *so that the infinitely long cylinder of greatest attraction will be an infinitely long rectangle, with its edge turned to the attracted point.*

If $m = 3$, we have $ax = x^2 + y^2$, the equation of a circle with the attracted point in its circumference.

If $m = 4$, the equation is $a^2x = (x^2 + y^2)^{\frac{3}{2}}$, which is Mr. PLAYFAIR's *curve of equal attraction.*

If we want the figure of the infinite cylinder of greatest attraction, when $m = 1$, we must have recourse to the last corollary; where we found

$$x\mathrm{L}\,(x^2 + y^2 + d^2) - x\mathrm{L}\,(x^2 + y^2) = C'.$$

This, when d is infinite gives $x\mathrm{L}.d^2 = C'$, or, $x = \text{const.}$, *the equation of a plane perpendicular to the axis of* x.

Cor. 4. If we would solve *Proposition* 34, but with this difference, that the force is now inversely as the mth power of the distance, and the density, in the generating rectangle uvv′u′, fig. 15, is, at any point, as its distance from rm or y; we need only put f (x) (given by the nature of the curve pr) for d, in the equation here found, and we get that of pr, in the case of

greatest attraction: viz.

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+f(x)^2)^{\frac{m-1}{2}}} + Cf(x)^2 = o.$$

Prop. 36.

To solve *Prop.* 32, the force being supposed inversely as the *m*th power of the distance, and the generating polygon being composed of triangles having such a law of density as that in the scholium to lemma 3.

By using the value found in that scholium, and proceeding, in other respects, as in the similar propositions already given, we find, for the equation of the curve touching the sides of the polygon,

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+(1+r^2)y^2)^{\frac{m-1}{2}}} + Cr^2y^2 = o.$$

Prop. 37.

Let *Prop.* 32 be yet once more resolved, on the supposition that the force is inversely as the *m*th power of the distance; and the density, in the triangles forming the generating polygon, either uniform, or as any function of x and T.

If we make use of the first value of A in lemma 3, we get, for the equation of the curve touching the sides of the polygon,

$$\frac{x}{(x^2+y^2)(x^2+(1+r^2)y^2)^{\frac{m-1}{2}}}\left\{1-(2-m).\frac{r^2y^2}{3(x^2+y^2)}+(2-m)(4-m)\right.$$
$$\left.\frac{r^4y^4}{3.5(x^2+y^2)^2} - \&c.\right\} + C = o.$$

When $r = o$, or the polygon becomes a circle, this equation is reduced to $\frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} + C = o$, as was found in another manner, in *Cor.* 1, *Prop.* 35.

If r is finite, the above expression will terminate when m is a whole positive even number; and consequently the guiding curve will then be algebraic. But, if m be amongst the numbers 5, 7, 9, 11, &c., we must use the other expression found in the lemma, and there arises, for the guiding curve, the transcendent equation

$$x\phi\,(x, m, y, ry) + Cry = o.$$

If $m = 1$, the equation is

$\frac{x}{\sqrt{x^2+y^2}} \times \text{arc}\left(\text{tang.} = \frac{ry}{\sqrt{x^2+y^2}}\right) + Cry = o$; and, finally, when $m = 3$,

$$\frac{x}{x^2+y^2} \times \frac{ry}{x^2+(1+r^2)y^2} + \frac{x}{(x^2+y^2)^{\frac{3}{2}}}\,\text{arc}\left(\text{tang.} = \frac{ry}{\sqrt{x^2+y^2}}\right) + Cry = o.$$

In like manner, might be solved *Prop.* 31 and 34, the force and density being as in Lemma 3, but this I leave to the reader.

Prop. 38.

The force being inversely as the mth power of the distance (where m is any whole positive number), and the density either uniform or any function of x and T,* the base of the infinitely long cylinder of greatest attraction has, for its equation,

$$\frac{x}{(x^2+y^2)^{\frac{m}{2}}} + C = o;$$

for it will appear from lemma 3, and its corollaries, that, whether m be odd or even (that is to say when it is any number in the series 1, 2, 3, 4, 5, &c.), the attraction of an infinite cylinder will be of the form

* What this means with respect to a cylinder, is shewn at the end of the scholium to *Prop.* 33; and with respect to a solid of revolution in *Prop.* 33.

$A = D \iint \frac{x f(x, T) \dot{T} \dot{x}}{(x^2 + T^2)^{\frac{m}{2}}}$, D being a function of m;

hence the truth of the proposition is manifest. And because the equation of the curve generating the solid of revolution of greatest attraction (on the same hypotheses of force and density) has been shewn to be $\frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} + C = o$, we have the following remarkable

Theorem.

m being any whole positive number, and the density either uniform or as any function of x and T, the same curve which, by revolving, generates the solid of revolution of greatest attraction, when the force is inversely as the mth power, shall be the base of the infinitely long cylinder of greatest attraction, when the force is inversely as the (m + 1th) power.

Numberless other interesting questions might be proposed, relating to solids of greatest attraction; for instance, we may inquire what must be the curve bounding the base of a cylinder of given mass and length so that it shall exercise the greatest action in a direction *parallel* to its axis.

But as this kind of inquiry proceeds exactly in the same way as the other (only we must use the attraction B, instead of A, in *Prop.* 1), it is unnecessary to lengthen a paper which has already been extended too far.

APPENDIX TO §. III.

Of the Attraction of an infinitely long Prism, whose Base is any right lined Figure whatever.

Prop. A.

Let the rectangle bb′ c′c, fig. 20, be the section or base of a prism, infinitely extended on both sides of it, and let the line psu bisect the opposite sides bb′, cc′ of the rectangle.

It is required to find the attraction of the infinitely long solid, on the point p, in the direction psu.

Let C be the centre of the rectangle, put $k = s\mathrm{C}$, $a = \mathrm{bs}$, $u = \mathrm{pC}$; draw rm perpendicular to $s\mathrm{Cu}$, and put $x = \mathrm{Cm}$. Now it appears, from *Cor.* 2, *Prop.* 1 of the paper (putting A for the required attraction) that

$\mathrm{A} = 4\int \dot{\overline{\mathrm{pm}}} \times \text{arc}\left(\text{tang.} = \frac{\mathrm{rm}}{\mathrm{pm}}\right) = 4\int \dot{x}\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right)$ $= 4x\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) - 4\int x\ \dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{a}{u+x}\right)$ the last term of which is $4\int \frac{ax\dot{x}}{(u+x)^2 + a^2}$; put $u + x = z$, $\dot{x} = \dot{z}$, $x = z - u$, and it becomes $4\int \frac{(az - au)\dot{z}}{a^2 + z^2}$, which is $4a\mathrm{L}.(a^2 + z^2)^{\frac{1}{2}} - 4u\ \text{arc}\left(\text{tang.} = \frac{z}{a}\right)$ so that

$\mathrm{A} = 4x\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) - 4u\ \text{arc}\left(\text{tang.} = \frac{u+x}{a}\right) + 4a\mathrm{L}.(a^2 + (u+x)^2)^{\frac{1}{2}}$, or $\mathrm{A} = 4(x+u)\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) - 2u\pi + 4a\mathrm{L}.(a^2 + (u+x)^2)^{\frac{1}{2}}$,

which fluent being taken from $x = -k$ to $x = k$ gives

$\mathrm{A} = 4(u+k)\ \text{arc}\left(\text{tang.} = \frac{a}{u+k}\right) - 4(u-k)\ \text{arc}\left(\text{tang.} = \frac{a}{u-k}\right) + 4a\mathrm{L}.(a^2 + (u+k^2)^{\frac{1}{2}} - 4a\mathrm{L}.(a^2 + (u-k^2)^{\frac{1}{2}}$.

If we choose to express this by the lines and angles of the figure (20), it is

$A = 4 \times \text{pu} \times \text{arc}, c\text{pu} - 4 \times \text{ps} \times \text{arc}, \text{bps} + 4 \times \text{bs} \times L\frac{\text{pc}}{\text{pb}}.$

Prop. B.

Let the section of the prism be an isosceles triangle; the attracted point p being in the line psm (fig. 21), which passes through the vertex s to the middle of the base r′p′.

Draw rm parallel to the base, and put $r =$ tang. rsm; call ps, u; sm, x; then rm $= rx$; and we have for the attraction of the infinite solid

$A = 4\int \dot{x}$ arc $\left(\text{tang.} = \frac{rx}{u+x}\right) = 4x$ arc $\left(\text{tang.} = \frac{rx}{u+x}\right) -$

$4\int \frac{x\left(\frac{r\dot{x}}{u+x}\right)}{1+\left(\frac{rx}{u+x}\right)^2}$; the last term is $-4\int \frac{rx\dot{x}(u+x) - rx^2\dot{x}}{(u+x)^2 + r^2x^2} = -$

$4\int \frac{rux\dot{x}}{(u+x)^2 + r^2x^2} = -4\int \frac{rux\dot{x}}{u^2 + 2ux + (1+r^2)x^2} = -\frac{4}{1+r^2}$

$\int \frac{rux\dot{x}}{\frac{u^2}{1+r^2} + \frac{2u}{1+r^2}x + x^2} = -\frac{4}{1+r^2}\int \frac{rux\dot{x}}{\left(x+\frac{u}{1+r^2}\right)^2 + \left(\frac{ur}{1+r^2}\right)^2} = -$

$\frac{4}{1+r^2}\int \frac{rux\dot{x}}{(x+\alpha)^2 + r^2\alpha^2}$, if we put $\alpha = \frac{u}{1+r^2}$. Make, moreover $x + \alpha$ $= z$, $x = z - \alpha$, $\dot{x} = \dot{z}$, and it becomes $-\frac{4}{1+r^2}\int \frac{(ruz - ru\alpha)\dot{z}}{z^2 + r^2\alpha^2}$, which fluent is

$$-\frac{4}{1+r^2}\left\{ruL.\sqrt{z^2 + r^2\alpha^2} - u \text{ arc } \left(\text{tang.} = \frac{z}{r\alpha}\right)\right\},$$

we have then at length

$A = 4x$ arc $\left(\text{tang.} = \frac{rx}{u+x}\right) - \frac{4}{1+r^2}\left\{ruL\sqrt{(x+\alpha)^2 + r^2\alpha^2} - u\right.$ arc $\left.\left(\text{tang.} = \frac{x+\alpha}{r\alpha}\right)\right\}$ + cor.

Cor. If the position of the attracting solid be reversed, as in fig. 22, call ps, u, and the attraction will be given by the same formula; only the fluent (if it begin at the point) must now

be taken from o to $-x$, instead of from o to x. A being a function of r, x, and u, may be represented by $2\phi\,(r, x, u)$. To correct the fluent, let sm (fig. 23) $=$ X, sm$'=x$, then, the attraction of the solid, whose base is the quadrilateral figure ρrr$'\rho'$, will be $2\phi\,(r, x, u) - 2\phi\,(r, \mathrm{X}, u)$.

In figure 24, call ps, u; sm, X; sm$'$, x. The action of the solid, whose base is ρrr$'\rho'$, is expressed by $2\phi\,(r, -x, u) - 2\phi\,(r, -\mathrm{X}, u)$.

In fig. 25, put ps $= u$, ps$' = u'$, sm $=$ s$'$m $= x$, tang. of rsm $= r$: the attraction of the solid, whose base is the rhombus srs$'\rho$, on a point p in the produced diameter of the section, is $2\phi\,(r, x, u) - 2\phi\,(r, o, u) + 2\phi\,(r, -x, u') - 2\phi\,(r, -o, u')$.

Prop. C.

Let fig. 26 represent the base or section of an infinitely long prism, and let this base be any right lined figure whatever, regular or irregular: from p, a point in the same plane, draw *any line* pq, cutting the base at s and m$'''$. It is required to find the action of the solid on the point p, in the direction pq.

From the angles r, r$'$, r$''$, r$'''$, &c. of the base, let fall the perpendiculars rm, r$'$m$'$, r$''$m$''$, r$'''$m$'''$, &c. on the line pq. Prolong the sides of the polygon till they meet pq at the points s, s$'$, s$''$, s$'''$, &c.

Put $u =$ ps, $u' =$ ps$'$, $u'' =$ ps$''$, $u''' =$ ps$'''$, &c.; and

$$x = \mathrm{sm}, \left\{\begin{matrix} x' = \mathrm{s'm'} \\ \mathrm{X}' = \mathrm{s'm} \end{matrix}\right\}, \left\{\begin{matrix} x'' = \mathrm{s''m''} \\ \mathrm{X}'' = \mathrm{s''m'} \end{matrix}\right\}, \left\{\begin{matrix} x''' = \mathrm{s'''m''} \\ \mathrm{X}''' = \mathrm{s'''m'''} \end{matrix}\right\}, \&c.$$

Also, let $r =$ tang. rsm, r$' =$ tang. r$'$s$'$m$'$, r$'' =$ tang. r$''$s$''$m$''$, $r''' =$ tang. r$'''$s$'''$m$'''$, &c. then it appears from the last proposition, that the attraction, of the upper half of the solid, is

expressed by

$$\begin{array}{l} \varphi(r,\ x,u) - \varphi(r,\ o,u) \\ + \varphi(r',\ x',u') - \varphi(r',\ X',u') \\ + \varphi(r'',\ x'',u'') - \varphi(r'',\ X'',u'') \\ + \varphi(r''',-x''',u''') - \varphi(r''',-X''',u''') \\ + \qquad \&c. \qquad - \qquad \&c. \end{array}$$

And in the same manner is found the attraction of the lower portion. If any part of the polygon, as pp', is parallel to pq, the attraction of that portion of the solid may be found by *Prop.* A.

Scholium to Prop. 25, page 273.

The following expression includes the attraction (on a point at the pole or vertex) of all this class of solids, where the generating plane is a regular polygon, and guiding curve a conic section: or where $y^2 = \alpha^2 (\beta x + \gamma x^2)$.

$A = 2n \left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right) \text{arc} \left(\text{tang.} = \frac{1}{rx} \sqrt{\mu x + \nu x^2}\right) - \left\{(n-2) x + \frac{n\beta\alpha^2}{1+\gamma\alpha^2}\right\} \pi + \varphi$

in which $\mu = \beta\alpha^2 (1 + r^2)$, $\nu = 1 + \gamma\alpha^2 (1 + r^2)$, and

$\varphi = \frac{2nr\beta\alpha^2}{\sqrt{\nu}\,(1+\gamma\alpha^2)} L \left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu} + 1}\right)$, or $= \frac{2nr\beta\alpha^2}{\sqrt{-\nu}\,(1+\gamma\alpha^2)}$ arc $\left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right)$, accordingly as ν is positive or negative.

Ex. 1. Let $\gamma = 0$, $\alpha = 1$, $y = \beta x$; in which case the solid is the polygonal parabolic conoid treated of in the proposition; and we have $\mu = \beta (1 + r^2)$, $\nu = 1$, whence

$$\mathrm{A} = 2n(x+\beta)\,\mathrm{arc}\left(\mathrm{tang.} = \frac{1}{rx}\sqrt{\beta(1+r^2)x+x^2}\right) - \left\{(n-2)\right.$$

$$\left. x + n\beta\right\}\pi + 2nr\beta\mathrm{L}\left(\frac{\sqrt{x}}{\sqrt{\beta(1+r^2)}} + \sqrt{\frac{x}{\beta(1+r^2)}+1}\right),$$

the same as was found before.

Ex. 2. Let $\alpha^2 = \frac{b^2}{a^2}$, $\beta = a$, $\gamma = -1$, $y^2 = \frac{b^2}{a^2}(ax - x^2)$: here the curve pr, fig. 15, is an ellipsis whose diameters are a and b, a being that which coincides with the axis pm. We have, in this case, $\mu = \frac{b^2}{a}(1+r^2)$, $\nu = 1 - \frac{b^2}{a^2}(1+r^2)$, and the attraction of a *polygonal spheroid*, on a point at its pole, is

$$\mathrm{A} = 2n\left(x+\frac{b^2a}{a^2-b^2}\right)\mathrm{arc}\left(\mathrm{tang.} = \frac{1}{rx}\sqrt{\frac{b^2}{a}(1+r^2)x+\left(1-\frac{b^2}{a^2}(1+r^2)\right)x^2}\right)$$

$$-\left\{(n-2)x+\frac{nb^2}{a^2-b^2}\right\}\pi + \phi,$$

where $\phi = \frac{2nra^2b^2}{(a^2-b^2)\sqrt{a^2-(1+r^2)b^2}}\,\mathrm{L}\left\{\frac{\sqrt{\left\{a^2-(1+r^2)b^2\right\}x}}{\sqrt{b^2a(1+r^2)}} + \right.$

$$\left.\sqrt{\frac{\left\{a^2-(1+r^2)b^2\right\}x}{b^2a(1+r^2)}+1}\right\},\ \text{or}$$

$$= \frac{2nra^2b^2}{(a^2-b^2)\sqrt{(1+r^2)b^2-a^2}}\,\mathrm{arc}\left(\mathrm{sine} = \sqrt{\frac{\left\{(1+r^2)b^2-a^2\right\}x}{b^2a(1+r^2)}}\right),$$

accordingly as $\frac{a^2}{b^2}$ is greater or less than $1+r^2$, or as $\frac{a}{b}$ is greater or less than the secant of half the angle formed at the centre of the generating polygon by one of its sides.

When $x = a$, the first arc in the above expression becomes simply $\mathrm{arc}\left(\mathrm{tang.} = \frac{1}{r}\right) = \frac{(n-2)\pi}{2n}$, and we have for the action of the *whole solid*, $\mathrm{A} = \psi - \frac{2ab^2}{a^2-b^2}\pi$, ψ representing ϕ after a has been put for x.

In like manner, may the action of the solid be found when

the guiding curve is an hyperbola; the only difference between that case, and the one we have just considered, being in the value of γ, which must be taken $+1$ instead of -1.

Scholium to Cor. 3, *Prop.* 27, page 278.

If the variable rectangle is given *in species*, and the touching curves are conic sections; that is, if

$$y^2 = \alpha^2(\beta x + \gamma x^2),\ y'^2 = \alpha'^2(\beta x + \gamma x^2),$$

we shall have, for the action of the generated solid, on a point at its vertex by *Prop.* 4,

$$A = 4\int\dot{x}\ \text{arc}\ \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,\alpha^2(\beta x + \gamma x^2)}\right) +$$

$$4\int\dot{x}\ \text{arc}\ \left(\text{tang.} = \frac{1}{r'x}\sqrt{x^2 + (1+r'^2)\,\alpha'^2(\beta x + \gamma x^2)}\right) - 2\pi x,$$

where $r = \frac{\alpha'}{\alpha}$, $r' = \frac{\alpha}{\alpha'}$; and by actually taking the fluent,

$$A = 4\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right)\text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{\mu x + \nu x^2}\right) - \frac{2\beta\alpha^2}{1+\gamma\alpha^2}\pi + \phi$$

$$+ 4\left(x + \frac{\beta\alpha'^2}{1+\gamma\alpha'^2}\right)\text{arc}\left(\text{tang.} = \frac{1}{r'x}\sqrt{\mu' x + \nu' x^2}\right) - \frac{2\beta\alpha'^2}{1+\gamma\alpha'^2}\pi + \phi'$$

$-2\pi x$, where $\mu = \beta\alpha^2(1+r^2)$, $\nu = 1 + \gamma\alpha^2(1+r^2)$, $\mu' = \beta\alpha'^2(1+r'^2)$, $\nu' = 1 + \gamma\alpha'^2(1+r'^2)$,

$$\phi = \frac{4r\beta\alpha^2}{\sqrt{\nu}\,(1+\gamma\alpha^2)}\,L\left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu} + 1}\right),\ \text{or} = \frac{4r\beta\alpha^2}{\sqrt{-\nu}\,(1+\gamma\alpha^2)}\ \text{arc}$$

$$\left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right)$$

according as ν is positive or negative,

$$\phi' = \frac{4r'\beta\alpha'^2}{\sqrt{\nu'}\,(1+\gamma\alpha'^2)}\,L\left(\frac{\sqrt{\nu' x}}{\sqrt{\mu'}} + \sqrt{\frac{\nu' x}{\mu'} + 1}\right),\ \text{or} = \frac{4r'\beta\alpha'^2}{\sqrt{-\nu'}\,(1+\gamma\alpha'^2)}\ \text{arc}$$

$\left(\text{sine} = \frac{\sqrt{-\nu' x}}{\sqrt{\mu'}}\right)$ as ν' is positive or negative.

If, in the preceding expression, we make r and α' infinite, and $r' = o$, it is reduced to

$A = 4\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right)$ arc $\left(\text{tang.} = \frac{\alpha}{x}\sqrt{\beta x + \gamma x^2}\right) - \frac{2\beta\alpha^2}{1+\gamma\alpha^2}\pi + \phi$

where $\phi = \frac{4\beta\alpha}{\sqrt{\gamma}\,(1+\gamma\alpha^2)}\, \mathrm{L}\left(\frac{\sqrt{\gamma x}}{\sqrt{\beta}}\right) + \sqrt{\frac{\gamma x}{\beta} + 1}\Big)$, or $= \frac{4\beta\alpha}{\sqrt{-\gamma}\,(1+\gamma\alpha^2)}$

arc $\left(\text{sine} = \frac{\sqrt{-\gamma x}}{\sqrt{\beta}}\right)$ as ν or γ is positive or negative.

This is *the action of an infinitely long cylinder on a point at the vertex of its transverse section, the equation of the said section being* $y^2 = \alpha^2(\beta x + \gamma x^2)$.

Ex. If the base, or transverse section, is an ellipsis, or if $y^2 = \frac{b^2}{a^2}(ax - x^2)$, we have $\alpha^2 = \frac{b^2}{a^2}$, $\beta = a$, $\gamma = -1$; and $A = 4\left(x + \frac{ab^2}{a^2-b^2}\right)$ arc $\left(\text{tang.} = \frac{b}{ax}\sqrt{ax - x^2}\right) + \frac{4a^2b}{a^2-b^2}$ arc $\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{a}}\right) - \frac{2ab^2}{a^2-b^2}\pi$. When $x = a$, this expression is reduced to

$$A = \frac{2ab}{a+b}\pi.$$

Scholium to Cor. 2, *Prop.* 30, page 281.

If we would have a general expression for the attraction of such solids as the one we considered in the proposition, when the guiding curve is any conic section, or when $y^2 = \alpha^2(\beta x + \gamma x^2)$, there arises at first (from the formula for the action of a rhombus)

$A = 4\int\dot{x}$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha^2(\beta x + \gamma x^2)}\right) +$

$4\int\dot{x}$ arc $\left(\text{tang.} = \frac{1}{r'x}\sqrt{x^2 + (1+r'^2)\alpha^2(\beta x + \gamma x^2)}\right) - 2\pi x$,

and by actually taking the fluent

$A = 4\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right)\Big\{$arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{\mu x + \nu x^2}\right) +$ arc $\left(\text{tang.} = \frac{1}{r'x}\sqrt{\mu' x + \nu' x^2}\right)\Big\} - \left(\frac{4\beta\alpha^2}{1+\gamma\alpha^2} + 2x\right)\pi + \phi + \phi'$,

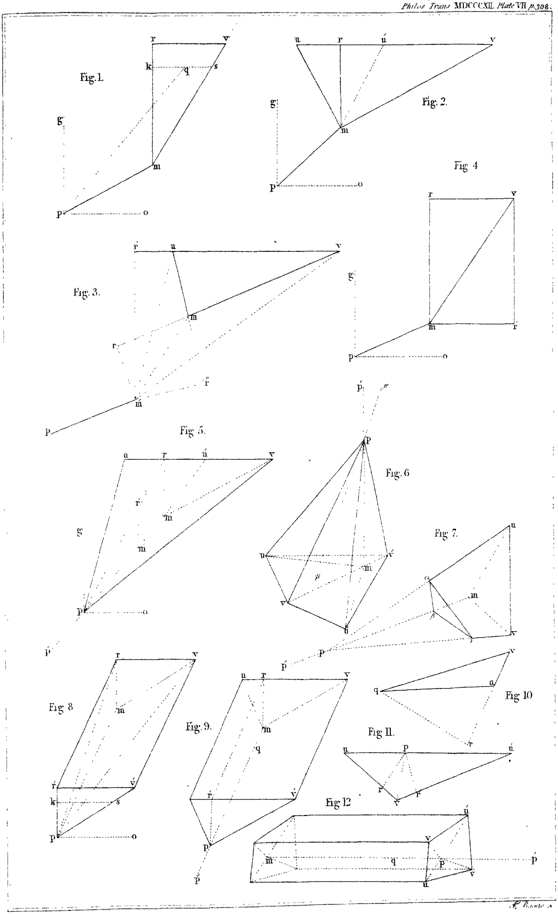


Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4
Fig. 5.
Fig. 6
Fig. 7.
Fig. 8
Fig. 9.
Fig. 10
Fig. 11.
Fig. 12

J. Basire sc.

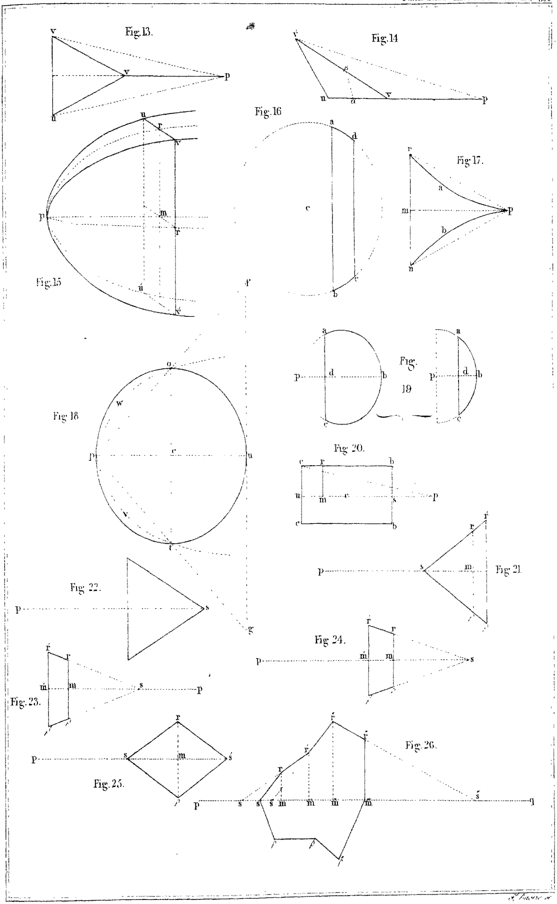


Fig. 13.
Fig. 14
Fig. 15
Fig. 16
Fig. 17.
Fig. 18
Fig. 19
Fig. 20.
Fig. 21.
Fig. 22.
Fig. 23.
Fig. 24.
Fig. 25.
Fig. 26.

J. Basire sc.

where $\mu = \beta\alpha^2 (1 + r^2)$, $\nu = 1 + \gamma\alpha^2 (1 + r^2)$, $\mu' = \beta\alpha^2 (1 + r'^2)$, $\nu' = 1 + \gamma\alpha^2 (1 + r'^2)$,

$$\phi = \frac{4r\beta\alpha^2}{\sqrt{\nu}\,(1+\gamma\alpha^2)} \text{L} \left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu} + 1}\right), \text{ or } = \frac{4r\beta\alpha^2}{\sqrt{-\nu}\,(1+\gamma\alpha^2)} \text{ arc}$$

$\left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right)$ as ν is positive or negative,

$$\phi' = \frac{4r'\beta\alpha^2}{\sqrt{\nu'}\,(1+\gamma\alpha^2)} \text{L} \left(\frac{\sqrt{\nu' x}}{\sqrt{\mu'}} + \sqrt{\frac{\nu' x}{\mu'} + 1}\right), \text{ or } = \frac{4r'\beta\alpha^2}{\sqrt{-\nu'}\,(1+\gamma\alpha^2)} \text{ arc}$$

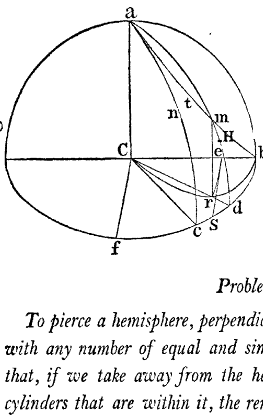
$\left(\text{sine } \frac{\sqrt{-\nu' x}}{\sqrt{\mu'}}\right)$ as ν' is positive or negative.

XV. *Of the Penetration of a Hemisphere by an indefinite Number of equal and similar Cylinders. By* Thomas Knight, *Esq. Communicated by Sir* Humphry Davy, *LL. D. Sec. R. S.*

Read March 19, 1812.

THE well known theorems of VIVIANI and BOSSUT, respecting certain portions of the surface and solidity of a hemisphere, form, together, a single case of the following problem; which is one of the most remarkable, for generality and simplicity of result, in the whole compass of geometry.

Fig. 1.



Problem.

To pierce a hemisphere, perpendicularly on the plane of its base, with any number of equal and similar cylinders; of such a kind, that, if we take away from the hemisphere those portions of the cylinders that are within it, the remaining part shall admit of an exact cubature: and if we take away, from the surface of the hemisphere, those portions cut out by the cylinders, the remaining surface shall admit of an exact quadrature.

Let fig. 1 represent the nearest half of the hemisphere, where a is the pole, bdcf a quadrant of the great circle form-

ing its base. From every point d, on this side of b, draw the radius dC to the centre of the hemisphere, and (if the number of cylinders is to be $2n$*) take the arc bs equal to n times the arc bd, draw se perpendicular to Cb, and with the centre C and radius Ce describe the arc er cutting Cd in r. Through all the points (r) thus found, draw the curve line brC, terminated at b and C, and it shall be half the base of one of the required cylinders.

It is, in the first place, evident, from the construction, that the half cylinder, whose base is beCrb, is contained between two planes CabC, CacC, making with each other the angle $bCc = \frac{90^\circ}{n}$; consequently the whole base of the hemisphere may be pierced by $2n$ such cylinders as this is the half of.

Let atmb be the intersection of the surfaces of the half cylinder and hemisphere; amd a great circle passing through a and d, and meeting atmb at m. Call the radius of the sphere r, Cr is the cosine of the arc bs to the radius r, by construction; it is also the cosine of the arc md to the same radius; therefore $md = bs = n \times bd$.

Put $bd = \phi$; $md = n \times \phi$; $d\varkappa = \psi$. Moreover, put A for the spherical space atmbdcna contained by the arcs anc, cdb and the curve atmb; and let S be the solidity of the portion of the hemisphere contained between the quadrant ancC and the surface (brCatmb) of the half cylinder. It is easy to see that

$$A = r^2 \iint \dot{\phi} \cos. \psi \times \dot{\psi},$$

$$S = \frac{r}{2} \iint \dot{\phi} \cos. \psi \times \cos. \psi \times \cos. \psi \, \dot{\psi}$$
$$- \frac{r^3}{2} \iint \dot{\phi} \cos. n\phi \times \cos. n\phi \times \cos. \psi \, \dot{\psi}.$$

* I do not intend $2n$ to represent an even number *only*, n may be $\frac{1}{2}$, or $\frac{3}{2}$, or $\frac{5}{2}$, &c. and $2n$ express any number whatever.

The fluents to be taken, first from $\psi = o$ to $\psi = n\phi$, and then from $\phi = o$, to $\phi = \frac{90^\circ}{n}$. The first operation gives

$$A = r^2 \int \phi \sin. n\phi,$$

$$S = \frac{r^3}{2} \int \phi \left\{ \frac{3}{4} \sin. n\phi + \frac{1}{12} \sin. 3n\phi \right\} - \frac{r^3}{2} \int \phi \sin. n\phi \cos.^2 n\phi,$$

and by the second we get

$$A = C - \frac{r^2}{n} \cos. n\phi$$

$$S = \frac{r^3}{2n} \left\{ \frac{\cos.^3 n\phi}{3} - \frac{3}{4} \cos. n\phi - \frac{1}{36} \cos. 3n\phi \right\} + C,$$

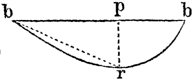
which fluents being taken from $n\phi = o$, to $n\phi = 90^\circ$, are $A = \frac{r^2}{n}$; $S = \frac{2}{9} \times \frac{r^3}{n}$; and if these are multiplied by $4n$, we have

$$A = 4r^2;\ S = \frac{8}{9} r^3;$$

for the whole that remains of the surface and solidity of the hemisphere after the subduction of the $2n$ cylinders. Thus A and S (for the whole hemisphere) do not depend on the number of the cylinders with which the penetration is made; *a most remarkable circumstance, seeing that amongst the bases of those cylinders are curves of an infinity of different kinds and orders.*

Let fig. 2 represent half the base of one of the cylinders;

Fig. 2.



Cb the radius of the hemisphere, C the centre. From r, any point in the curve, let fall the perpendicular rp on the axis; call Cp, x; rp, y.

By construction, $Cr = \sqrt{x^2 + y^2} = r \cos. n \,.\, bCr$; now the cosine of the simple arc bCr is $\frac{x}{\sqrt{x^2+y^2}}$, which being put in the trigonometrical expression for the cosine of the multiple arc

in terms of the cosine of the simple one, we have, for the equation of the curve brC.

When $n = 1$, $\sqrt{x^2 + y^2} = \frac{rx}{\sqrt{x^2+y^2}}$ or $x^2 + y^2 = rx$, the equation of a circle.

When $n = 2$, $\sqrt{x^2 + y^2} = \frac{2rx^2}{x^2+y^2} - r$ or $(x^2 + y^2)^3 = r^2 (x^2 - y^2)^2$; and in general the curve will be algebraic when n is any whole number.

XVI. *On the Motions of the Tendrils of Plants.* *By* Thomas Andrew Knight, *Esq. F. R. S. In a Letter to the Right Hon. Sir* Joseph Banks, *Bart. K. B. P. R. S.*

Read May 4, 1812.

My Dear Sir,

The motions of the tendrils of plants, and the efforts they apparently make to approach and attach themselves to contiguous objects, have been supposed by many naturalists to originate in some degrees of sensation and perception: and though other naturalists have rejected this hypothesis, few, or no experiments have been made by them to ascertain with what propriety the various motions of tendrils, of different kinds, can be attributed to peculiarity of organization, and the operation of external causes. I was consequently induced, during the last summer, to employ a considerable portion of time to watch the motions of the tendrils of different species of plants; and I have now the pleasure to address to you an account of the observations I was enabled to make.

The plants selected were the Virginia creeper (the ampelopsis quinquefolia of Michaux,) the ivy, and the common vine and pea.

A plant of the ampelopsis, which grew in a garden pot, was removed to a forcing house in the end of May, and a single shoot from it was made to grow perpendicularly upwards, by being supported in that position by a very slender bar of wood, to which it was bound. The plant was placed in the middle

of the house, and was fully exposed to the sun; and every object around it was removed far beyond the reach of its tendrils. Thus circumstanced, its tendrils, as soon as they were nearly full grown, all pointed towards the north, or back wall, which was distant about eight feet: but not meeting with any thing in that direction, to which they could attach themselves, they declined gradually towards the ground, and ultimately attached themselves to the stem beneath, and the slender bar of wood.

A plant of the same species was placed at the east end of the house, near the glass, and was in some measure skreened from the perpendicular light; when its tendrils pointed towards the west, or centre of the house, as those under the preceding circumstances had pointed towards the north and back wall. This plant was removed to the west end of the house, and exposed to the evening sun, being skreened, as in the preceding case, from the perpendicular light; and its tendrils, within a few hours, changed their direction, and again pointed to the centre of the house, which was partially covered with vines. This plant was then removed to the centre of the house, and fully exposed to the perpendicular light, and to the sun; and a piece of dark-coloured paper was placed upon one side of it just within the reach of its tendrils; and to this substance they soon appeared to be strongly attracted. The paper was then placed upon the opposite side, under similar circumstances, and there it was soon followed by the tendrils. It was then removed, and a piece of plate glass was substituted; but to this substance the tendrils did not indicate any disposition to approach. The position of the glass was then changed, and care was taken to adjust its surface to the varying

position of the sun, so that the light reflected might continue to strike the tendrils; which then receded from the glass, and appeared to be strongly repulsed by it.

The tendrils of the ampelopsis very closely resemble those of the vine, in their internal organization, and in originating from the alburnous substance of the plant; and in being, under certain circumstances, convertible into fruit stalks. The claws, or claspers, of the ivy, to experiments upon which I shall now proceed, appear to be cortical protrusions only; but to be capable (I have reason to believe) of becoming perfect roots, under favourable circumstances. Experiments, in every respect very nearly similar to the preceding, were made upon this plant; but I found it necessary to place the different substances, to which I proposed that the claws should attempt to attach themselves, almost in contact with the stems of the plants. I observed that the claws of this plant evaded the light, just as the tendrils of the ampelopsis had done; and that they sprang only from such parts of the stems as were fully, or partially, shaded.

A seedling plant of the peach tree, and one of the ampelopsis and ivy, were placed nearly in the centre of the house, and under similar circumstances; except that supports, formed of very slender bars of wood, about four inches high, were applied to the ampelopsis, and ivy. The peach tree continued to grow nearly perpendicularly, with a slight inclination towards the front and south side of the house, whilst the stems of the ampelopsis and ivy, as soon as they exceeded the height of their supports, inclined many points from the perpendicular line, in the opposite direction.

It appears therefore that not only the tendrils and claws of

these creeping dependent plants, but that their stems also, are made to recede from light, and to press against the opake bodies, which nature intended to support and protect them.

M. Decandole, I believe, first observed that the succulent shoots of trees and herbaceous plants, which do not depend upon others for support, are bent towards the point from which they receive light, by the contraction of the cellular substance of their bark, upon that side, and I believe his opinion to be perfectly well founded. The operation of light upon the tendrils and stems of the ampelopsis and ivy appears to produce diametrically opposite effects, and to occasion an extension of the cellular bark, wherever that is exposed to its influence; and this circumstance affords, I think, a satisfactory explanation why these plants appear to seek and approach contiguous opake objects, just as they would do, if they were conscious of their own feebleness, and of power in the objects, to which they approach, to afford them support and protection.

The tendril of the vine, as I have already stated, is internally similar to that of the ampelopsis, though its external form, and mode of attaching itself, by twining round any slender body, are very different. Some young plants of this species, which had been raised in pots in the preceding year, and had been headed down to a single bud, were placed in a forcing-house, with the plants I have already mentioned; and the shoots from these were bound to slender bars of wood, and trained perpendicularly upwards. Their tendrils, like those of the ampelopsis, when first emitted, pointed upwards; but they gradually formed an increasing angle with the stems, and ultimately pointed perpendicularly downwards; no object

having presented itself to which they could attach themselves.

Other plants of the vine, under similar circumstances were trained horizontally; when their tendrils gradually descended beneath their stems, with which they ultimately stood very nearly at right angles.

A third set of plants were trained almost perpendicularly downwards; but with an inclination of a few degrees towards the north; and the tendrils of these permanently retained very nearly their first position, relatively to their stems; whence it appears that these organs, like the tendrils of the ampelopsis, and the claws of the ivy, are to a great extent under the control of light.

A few other plants of the same species were trained in each of the preceding methods; but proper objects were placed, in different situations, near them, with which their tendrils might come into contact; and I was by these means afforded an opportunity of observing, with accuracy, the difference between the motions of these and those of the ampelopsis, under similar circumstances. The latter almost immediately receded from light, by whatever means that was made to operate upon them; and they did not subsequently shew any disposition to approach the points, from which they once receded. The tendrils of the vine, on the contrary, varied their positions in every period of the day, and after, returned again during the night to the situations they had occupied in the preceding morning; and they did not so immediately, or so regularly, bend towards the shade of contiguous objects. But as the tendrils of this plant, like those of the ampelopsis, spring alternately from each side of the stem, and as one point only in three is with-

out a tendril, and as each tendril separates into two divisions they do not often fail to come into contact with any object within their reach; and the effects of contact upon the tendril are almost immediately visible. It is made to bend towards the body it touches, and, if that body be slender, to attach itself firmly by twining round it, in obedience to causes which I shall endeavour to point out.

The tendril of the vine, in its internal organization, is apparently similar to the young succulent shoot, and leaf-stalk, of the same plant; and it is as abundantly provided with vessels, or passages, for the sap; and I have proved that it is alike capable of feeding a succulent shoot, or a leaf, when grafted upon it. It appears therefore, I conceive, not improbable, that a considerable quantity of the moving fluid of the plant passes through its tendrils; and that there is a close connection between its vascular structure and its motions.

I have proved in the Philosophical Transactions of 1806, that centrifugal force, by operating upon the elongating plumules of germinating seeds, occasions an increased growth and extension upon the external sides of the young stems, and that gravitation produces correspondent effects; probably by occasioning the presence of a larger portion of the fluid organizable matter of the plant upon the one side, than upon the other. The external pressure of any body upon one side of a tendril will probably drive this fluid from one side of the tendril, which will consequently contract, to the opposite side, which will expand; and the tendril will thence be compelled to bend round a slender bar of wood or metal, just as the stems of germinating seeds are made to bend upwards, and to raise the cotyledons out of the ground; and in support of this

conclusion I shall observe, that the sides of the tendrils, where in contact with the substance they embraced, were compressed and flattened.

The actions of the tendrils of the pea were so perfectly similar to those of the vine, when they came into contact with any body, that I need not trouble you with the observations I made upon that plant. An increased extension of the cellular substance of the bark upon one side of the tendrils, and a correspondent contraction upon the opposite side, occasioned by the operation of light, or the partial pressure of a body in contact, appeared in every case, which has come under my observation, the obvious cause of the motions of tendrils; and therefore, in conformity with the conclusions I drew in my last memoir, respecting the growth of roots, I shall venture to infer, that they are the result of pure necessity only, uninfluenced by any degrees of sensation, or intellectual powers.

I am, my dear Sir,

with much regard, &c.

THO. ANDREW KNIGHT.

Downton, April 27, 1812.

To the Right Hon. Sir JOSEPH BANKS, Bart. K. B.

XVII. *Observations on the Measurement of three Degrees of the Meridian conducted in England by Lieut. Col.* William Mudge. *By Don* Joseph Rodriguez. *Communicated by* Joseph de Mendoza Rios, *Esq. F. R. S.*

Read June 4, 1812.

THE determination of the figure and magnitude of the earth has at all times excited the curiosity of mankind, and the history of the several attempts made by astronomers to solve this problem might be traced to the most remote antiquity. But the details of the methods pursued by the ancients on this subject being extremely vague, and their results expressed in measures of which we do not know the relation to our own, in fact give us very little assistance in learning either the figure or dimensions of our globe.

It was not till the revival of science in Europe that the two great philosophers, HUYGHENS and NEWTON, first engaged in the consideration of this question, and reduced to the known laws of mechanics, the principles on which the figure of the earth should be determined.

They demonstrated that the rotatory motion should occasion differences in the force of gravity in different latitudes, and consequently that parts of the earth in the neghbourhood of the equator should be more elevated than those near the poles.

The most simple hypothesis, which first presented itself to

their imagination, was that which supposed the earth to be throughout composed of the same kind of matter, and its surface that of a spheroid generated by revolution round its axis. This hypothesis, adopted by NEWTON only as an approximation to the truth, is, in fact, perfectly consistent with the equilibrium to which particles in a state of paste, or of tardy fluidity, would arrive in a short time after their present motion was impressed; and the eccentricity derived from this hypothesis is at least not very remote from that which actually obtains in the present state of consistence and stability which the earth has since acquired.

But the homogeneity of the matter, of which the earth consists, is at variance with all geological observations, which prove evidently that at least 5000 toises of the exterior crust is formed of an immense mass of heterogeneous matters varying in density from each other; and upon the supposition of a state of fluidity of the whole, it should follow that the strata should successively increase in density from the surface towards the centre, that the more dense would accordingly be subjected to less of centrifugal force, and consequently that the spheroidical form resulting from this cause would be less eccentric than would arise from a state of perfect homogeneity.

The most simple, as well as the most effectual means of verifying the hypothesis respecting the figure of the earth, is to measure in the two hemispheres several arcs of its meridians in different latitudes, at some distance from each other. On this subject it must be allowed, that the Academy of Sciences at Paris set the example, in giving the original impulse to the undertaking, and not only commenced, but put

in execution those parts of the plan which were most difficult and most decisive.

The results of the first measurements made of different arcs on the meridian of different parts of the world, were found to be perfectly conformable to the expectations of HUYGHENS and of NEWTON, and also with experiments made on the vibration of the pendulum in different latitudes; and they left no doubt that the earth was in fact flattened at the poles; establishing thereby one point extremely interesting in natural philosophy.

These results, however, did not correspond with sufficient accuracy for ascertaining with precision the degree of eccentricity, or even the general dimensions of the earth, as might naturally be expected when we consider the necessary imperfection of the means then employed in these operations, and the great difficulties that are to be encountered.

For the purpose of making a nearer approximation to the true dimensions of the earth, and of verifying former measurements, it is necessary in some instances to repeat them, and also to make others in different situations, which may be expected to be improved in proportion to the progress that is made in the means of perfecting the several departments of science.

At the commencement of the French revolution, men of science took advantage of the general impulse which the human mind received in favour of every species of innovation, or change, and they proposed making a new measurement of an arc of the meridian in France, for the purpose of establishing a new system of weights and measures, which should be permanent, as being founded on the nature of things.

A commission, composed of some of the most distinguished members of the Academy of Sciences, was charged to form the plan of these operations, which were to serve as the basis of the new system. They invented new instruments, new methods, new formulæ, and in short almost the whole of this important undertaking consisted of something new in science.

Two celebrated astronomers, DELAMBRE and MECHAIN, were engaged to perform the astronomical and geodetical observations, and these they continued as far as Barcelona in Spain. The details of their operations, observations, and calculations, were subsequently examined by a committee of men of science, many of whom were foreigners collected at Paris, who confirmed their results, and by the sanction of such an union of talents, gave such a degree of credit and authenticity to their conclusions as could scarcely be acquired by other means.

Since that time, in the year 1806, Messrs. BIOT and ARAGO, members of the National Institute, were sent into Spain for the express purpose of carrying on the same course of operations still further southward, from Barcelona as far as Formentera, the southernmost of the Balearic islands. Fortunately this last undertaking, which forms a most satisfactory supplement to the former, was completed by the month of May, 1808, at a period when political circumstances would not admit of any further operations being pursued, as a means of verifying the results, by measuring a base which should be independent of those formerly obtained in France.

In the year 1801, the Swedish Academy of Sciences, encouraged by the success of the operations conducted in France, sent also three of its members into Lapland, to verify their former measurement taken in 1736, by new methods, and by

the use of new instruments, similar to those which had recently been used in France, and of which the National Institute made a handsome present to the Swedish Academy. The results of this new undertaking, which terminated in 1803, were drawn up by M. SVANBERG, and are highly interesting, by their exactness, by the perspicuity of the details, and even a certain degree of novelty given to the subject by the arrangement adopted by the learned author M. SVANBERG.

These new measures were found to confirm, in a remarkable manner, the general results of those which had preceded, and gave very nearly the same proportion for the eccentricity and other dimensions of the globe, so that there would not have remained the smallest doubt respecting the figure of the earth being flattened at the poles, had there not been a fourth measurement performed in England at the same time as that undertaken in Lapland, the results of which were entirely the reverse. This measurement, which comprised an arc of 2° 50′, was undertaken by Lieut. Col. MUDGE, Fellow of the Royal Society, with instruments of the most perfect construction that had ever yet been finished by any artist, contrived and executed for that express purpose, by the celebrated RAMSDEN. The details of the observations and other operations of Lieut. Col. MUDGE, may be seen in the volume of the Philosophical Transactions for the year 1803; and one cannot but admire the beauty and perfection of the instruments employed by that skilful observer, as well as the scrupulous care bestowed on every part of the service in which he was engaged. Bengal lights were employed on this occasion, as objects at the several stations, and their position appears to have been determined with the utmost precision by the theodolite of RAMSDEN, which

reduces all angles to the plane of the horizon, and with such a degree of correctness, that the error in the sum of the three angles of any triangle is scarcely, in any instance, found to exceed three seconds of a degree, and in general not more than a small fraction of a second.

Accordingly the geodetical observations were conducted with a degree of exactness, which hardly can be exceeded; and even if we suppose for a moment, that the chains made use of in the measurement of the bases may not admit of equal precision with the rods of platina employed in France, nevertheless, the degree of care employed in their construction, in the mode of using them, and the pains taken to verify their measures was such, that no error that can have occurred in the length of the base, could make any perceptible difference in the sides of the series of triangles, of which the whole extent does not amount to so much as three degrees.

Nevertheless, the results deduced by the author, from this measure alone, would lead to the supposition that the earth, instead of being flattened at the poles, is in fact more elevated at that part than at the equator, or at least, that its surface is not that of a regular solid. For the measures of different degrees on the meridian, as reduced by Lieut. Col. Mudge, increase progressively toward the equator.

The following table of the different measures of a degree in fathoms is given by the author in his Memoir.

Latitude.	
52° 50′ 30″	60766
52 38 56	60769
52 28 6	60794
52 2 20	60820

Latitude.	
51° 51′ 4″	60849
51 25 18	60864
51 13 18	60890
51 2 54	60884

The singularity of these results excites a suspicion of some incorrectness in the observations themselves, or in the method of calculating from them. The author has not informed us in his Memoir, what were the formulæ which he employed in the computations of the meridian; but one sees, by the arrangement of his materials, that he made use of the method of the perpendiculars without regard to the convergence of the meridians; and although this method is not rigorously exact, it can make but a very few fathoms more in the total arc, and will have very little effect on the magnitude of each degree. It is therefore a more probable supposition, that, if any errors exist, they have occurred in the astronomical observations. But it is scarcely possible to determine the amount of the errors, or in what part of the arc they may have occurred, excepting by direct and rigorous computation of the geodetical measurement. I have therefore been obliged to have recourse to calculations, which I have conducted according to the method and formulæ invented and published by M. Delambre.

The means generally employed for finding the extent of a degree of the meridian, consists in dividing the length of the total arc in fathoms, by the number of degrees and parts of a degree deduced from observations of the stars; but if these observations are affected by any error, arising from unsteadiness of the instrument, from partial attractions, or from any other accidental causes, then the degrees of the meridian will

be affected, without a possibility of discovering such an error in this mode of operating. It is consequently necessary, in such a case, to employ some other method, which may serve as a means of verifying the observations themselves, of detecting their errors, if there be any, or at least of shewing their probable limits.

My object therefore is to communicate the result of calculations that I have made, from the data published by Lieut. Col. MUDGE in the Philosophical Transactions; and I hope to make it appear, that the magnitude of a degree of the meridian, corresponding to the mean latitude of the arc measured by this skilful observer, corresponds very exactly with the results of those other measurements that have been above noticed.

In M. DELAMBRE's method nothing is wanting but the spherical angles, that is to say, the horizontal angles observed, corrected for spherical error. Moreover, for our purpose, we have no occasion for the numerical value of the sides of the series of triangles, but only for their logarithms. Thus the logarithm of the base measured at Clifton, as an arc gives us that of its sine in feet or in fathoms, so that by means of this latter logarithm, and the spherical angles of the series of triangles, we obtain at once, and as easily as in plane trigonometry, the logarithms of the sines of all their sides in fathoms.

After this, it is extremely easy to convert them into logarithms of chords or of arcs, for the purpose of applying them to the computation of the arcs on the meridian or azimuths. I give the preference to taking the logarithms of the sides as arcs, because the computations become in that case much more simple and expeditious.

Near to Clifton, which is the northern extremity of the arc,

in a situation elevated 35 feet above the level of the sea, a base was measured of 26342,7 feet in length, the chains being supposed at the temperature of 62° FAHRENHEIT, or $13\frac{1}{3}°$ REAUMUR.

For reducing this base to toises, we have the proportion of the English foot to that of France, as 4 : 4,263, so that if p be taken to express the fractional part of the French foot, corresponding to English measure, then log. $p = 9{,}97234{,}46587$,

and then log. of $26{,}342{,}7 = 4{,}42066{,}02860$,

and hence the log. of the base in toises will be found equal to 3,61485,36943, and the number of toises corresponding is 4119,5 taken at the same temperature, which corresponds to $16\frac{2}{3}°$ of the centigrade thermometer.

This base we must consider as an arc of a circle, and it is easy to reduce it to the sine of the same arc, according to the method given in a note at the end of this memoir. The logarithm of the *sine* of the base in toises is found to be 3,61485,35800.

With this quantity as base, and by means of the spherical triangles given by Lieut. Col. MUDGE in his paper, I have found the logarithmic sines in toises of all the sides of his series of triangles, and have subsequently reduced them to logarithmic arcs of the same, which enable me to complete the rest of the calculation. With these we may compute any portions of the meridian, or successive intervals of different stations expressed in toises, and in parts of the circle, or their respective azimuths, having regard always to the relative convergence of different meridians.

The author has made observations for determining the latitude of the two extremities of his arc, and has also determined

the azimuths of the exterior sides in his series of triangles by means of the greatest elongation of the pole star.

In the calculations that I have made, I began at Clifton in Yorkshire, the northern extremity of the arc, and for this purpose the following are the data furnished by Lieut. Col. MUDGE.

Latitude of Clifton reduced to the centre of the station 53° 27′ 36,″62.

Azimuth of Gringley, seen from Clifton, and reckoned from the north toward the west 256° 17′ 25″.

Azimuth of Heathersedge, seen from Clifton, and reckoned in the same direction 118° 8′ 8″,81.

With these data, and the two tables of spherical triangles, and the logarithms of their sides expressed in arcs, the intervals between Clifton and the two stations Gringley and Heathersedge were found in toises and in seconds of a degree, as well as all the corrections to be made on the first azimuths increased by 180°, as azimuths of Clifton seen on the horizon at these latter places.

The same process was continued for the following stations in succession, all the way to Dunnose in the Isle of Wight, which is the southernmost extremity of the series.

In this manner we have the latitudes and azimuths of each station, by means of two or three preceding stations, and consequently we have a verification of all the calculations that have been before made by Lieut. Col. MUDGE.

The results of my calculations are contained in the two following tables.

First Table of Distances in Toises and in Seconds of a Degree on the Meridian, comprised between the westerly Stations in the Series of Triangles.

Names of the Stations.	Arcs in Toises.	Arcs in Seconds.
Clifton -	0,0	0,0
Heathersedge	6834,324	430,9928
Orpit -	15818,489	997,5928
Castlering -	19801,1934	1248,8226
Corley -	14295,384	901,6207
Epwell -	22327,008	1408,2543
Stow -	9555,479	602,7284
Whitehorse	18799,645	1185,8656
Highclere -	14990,567	945,6354
Dean Hill -	16105,614	1016,0180
Dunnose -	23529,886	1484,4531
Sum total -	162057,5437	10221,9837

Second Table of successive Intervals between the Eastern Stations.

Names of the Stations.	Arcs in Toises.	Arcs in Seconds.
Clifton -	0,0	0,0
Gringley -	2809,105	177,149
Sutton -	10838,816	1061,931
Holland Hill	4681,190	295,2251
Bardon Hill -	18092,261	1141,0462
Arbury Hill	27956,417	1763,2683
Brill -	22374,106	1411,2769
Nuffield -	14350,3834	905,2155
Bagshot -	12137,933	765,6822
Hindhead -	14449,2027	911,5140
Butser Hill -	7853,644	495,4551
Dunnose -	20514,036	1294,1974
Sum total -	162057,0941	10221,9607

Now if we take the arithmetic mean of the sums contained in the two tables, we have for measures of the entire arc, comprised between the stations of Clifton and Dunnose, the following quantities 162057,32 toises, and 10221,972 seconds of a degree, or 2° 50′ 21″,972. By dividing the former of these by the second, we get the measure of a degree, corresponding to the mean latitude of the whole arc, equal to 57073,74 toises, or 60826,34 fathoms, at the temperature of $16\frac{2}{3}°$ of the centigrade thermometer, the latitude being 52° 2′ 20″.

The station at Arbury Hill happens to be very nearly in the meridian of Clifton and Dunnose, and divides the interval between them into nearly equal parts. The measures of that part of the arc, which lies between Arbury and Dunnose, is by the tables 91679,47 toises, and 9783″,34 seconds, or 1° 36′ 23″,34 of the common division of the circle. The mean latitude of the arc is 51° 25′ 21″. And the measure of 1 degree corresponding to it is 57068,41 toises.

In the same manner the measure of the arc comprised between Arbury Hill and the northern extremity at Clifton, is 70377,85 toises, and 4438,63 seconds, or 1° 13′ 58″,63. Its mean latitude is 52° 50′ 32″. And we have for one degree of the meridian, corresponding to this latitude, 57080,70 toises.

Hence, if we divide the entire arc into two equal parts, we deduce the following values of a degree corresponding to the middle of the whole and of its parts.

Latitudes.	
51° 25′ 20″	57068
52 2 20	57074
52 50 30	57081

These values are, as appears, perfectly in conformity with the theory, and with the results of other measures that have been taken in different parts of the northern hemisphere; but, in order to place that agreement in a more distinct point of view, I shall show how nearly these estimates agree with the elliptic hypothesis, by comparing them with those measures of a degree, on which we can place the greatest reliance for exactness.

Now, if we compare the results of these calculations with those deduced by Lieut. Col. MUDGE from his observations, we shall see the probable source of those *errors*, which it appears to me have led him to false conclusions. It has already been observed, that the station at Arbury Hill divides the whole arc into two parts nearly equal, and that it is also nearly in the meridian of the two extremities at Dunnose and Clifton. It was, in all probability, this circumstance which determined the author to observe the latitude of Arbury Hill, as he would then have two partial arcs independent of the whole and of each other.

For determining the angular extent of these arcs, Lieut. Col. MUDGE observed the zenith distances of several stars on the meridian above the pole, by means of a large zenith sector constructed by RAMSDEN, with the same pains that he had bestowed upon the theodolite. Lieut. Col. MUDGE paid all possible attention, and took all such precautions as might naturally be expected from an observer of his experience and address. Nevertheless the results of his observations made on different stars, differ no less than 4 seconds from each other. But, by taking a mean of all, the dimensions of the three arcs reduced to the centre at each station are as follows.

Between Clifton and Dunnose	2° 50′ 23″,35
Clifton and Arbury	1 14 3 ,40
Arbury and Dunnose	1 36 19 ,95

The extent of the first arc, in linear measure, is 1036339½ feet English, and when this is reduced to toises, we have for the lengths of the three arcs from Lieut. Col. Mudge's measures,

From Clifton to Dunnose	162067,3
Clifton to Arbury	70380,2
Arbury to Dunnose	91687,1

These last values exceed those resulting from my computations, the first by 10 toises, the second by 2, the third by 8 toises; and these differences arise from the convergence of the meridians, which the author thought might safely be neglected, and in fact it does not make a difference that is perceptible in the value of a degree upon the meridian. For the difference of 8 toises, in the distance between Dunnose and Arbury, makes but 5 toises difference in the value of a degree upon that arc, and the difference of 10 in the whole distance from Dunnose to Clifton, makes 3½ in the measure of each degree on that arc. So that, as far as this source of disagreement is concerned, the author's results and mine would not be found to differ materially from each other.

But, if we attend to the angular dimensions of the several arcs, as deduced from observation and from calculation, these will not be found to agree so nearly.

The following table will shew the differences in each instance.

Clifton and Dunnose $\begin{cases} 2^\circ\ 50'\ 23'',35 & \text{observed} \\ 2\ \ \ 50\ \ \ 21\ \ ,97 & \text{calculated} \end{cases}$

Difference $+\ 1\ ,38$

Clifton and Arbury $\begin{cases} 1^\circ\ 14'\ \ 3'',40 & \text{observed} \\ 1\ \ \ 13\ \ \ 58\ \ ,63 & \text{calculated} \end{cases}$

Difference $+\ 4\ ,77$

Arbury and Dunnose $\begin{cases} 1^\circ\ 36'\ 19'',95 & \text{observed} \\ 1\ \ \ 36\ \ \ 23\ \ ,34 & \text{calculated} \end{cases}$

Difference $-\ 3\ ,39$

These differences are really considerable, and are capable of producing important errors in the results dependent on them.

In the first place we see, that the southernmost arc between Dunnose and Arbury is smaller than it would appear by computation, by as much as 3″,4, and when this deficiency is combined with an excess of 8 toises in the linear dimensions of the same arc, it makes as much as 40 toises difference in the estimated length of a degree. The reverse of this occurs in the northern portion of the arc comprised between Clifton and Arbury Hill. This is larger than it ought to be by 4″,77, and hence the value of a degree on the meridian turns out too small by about 62 toises in its linear dimensions. Fortunately, however, the excess of the total arc is extremely small, as it does not exceed 1″,38, so as to make but 5 or 6 toises difference in the length of a degree observed on the meridian, and corresponding to the mean latitude of the arc examined.

From what has been above stated, it seems almost beyond a doubt that it is to errors in the observations of latitude, that the appearance of progressive augmentation of degrees towards the equator, as represented by Lieut. Col. MUDGE in his paper, are to be ascribed, and that it is especially at the intermediate station at Arbury Hill, that the observations of the stars are erroneous nearly 5 seconds, notwithstanding the goodness of the instruments, and the skill and care of the observer. But, before I insist farther on this head, I will answer one objection that may be made to the principles of the method that I have pursued in this Memoir.

Those astronomers, who have hitherto undertaken the measurement of degrees of the meridian, have deduced their measures by simply dividing the linear extent by the number of degrees and minutes found by observation of the fixed stars taken at the two extremities of the arc. This is indeed the most simple that can be adopted; and it has the advantage of being independent of the elliptic figure of the earth, especially in arcs of small extent. The elements dependent on this figure, are too uncertain to be employed in calculating the angular intervals in the short distances between successive stations, even as a means of verification, without risk of committing greater errors than those to which astronomical observations can be liable. Accordingly one cannot safely make any use of it in cases where great accuracy is required.

I must admit the justness of this objection, and must therefore shew the extent to which it really applies to the present subject.

In the first place, I may suppose, that in consequence of some fault in the instrument, with respect to vertical position,

construction, or some accidental derangement, there is an error of some seconds in the observations of the fixed stars. How is this to be discovered? This is not to be done by comparing the value of a degree on the meridian, as deduced from these observations, with the results of other measurements in distant parts of the globe. For if we find that these degrees so taken do not agree in giving the same ellipsoid, we are not to attribute all the differences to irregularities of the earth, without supposing any error on the part of the observer, of his instrument, or of other means employed in his survey.

But this, in fact, is what has generally been done. It must, however, be acknowledged, that the majority of observers have not been in fault, as they could do nothing better; but too much reliance has been placed on the goodness of their instruments, their means, and other circumstances. It is true that irregularities of the earth and local attractions may occasion considerable discrepancies which are even inevitable; but before we decide that these are the real source of disagreement, we ought carefully to ascertain that there are no others.

But to return to our subject, of the English measurement. If the uncertainty which yet subsists, with respect to the exact figure of the earth and its dimensions, occasions some small errors in the calculation of the series of triangles, the sum of these errors will be found in the estimate of the entire arc, and will increase in proportion to the extent of the arc measured. Now, in the English measurement, we find exactly the reverse of this. For the difference between the results of calculation and observation is only $1'',38$ on the whole arc; but is even as high as $4'',77$ on one of the smaller arcs. So that, whatever error we may suppose to have been introduced

into the calculation by assuming a false estimate of the spheroidity of the earth, or of other elements employed in the calculation, it is very evident that the zenith distances of stars taken at Arbury Hill are affected by some considerable error, wholly independent of these elements.

It was not till the date of the measurement of the meridian in France, that M. DELAMBRE published and explained, with admirable perspicuity and elegance, all the formulæ and methods relative to the calculation of spheroids, and put it in the power of astronomers in general to make use of the elliptic elements in verifying the results of their observations. In the present state of science these elements are well known, and the errors that can arise from any uncertainty in them, are not so considerable as is generally supposed. The oblateness and the diameter at the equator are the only elements wanting in the calculation; for the purpose of seeing what effect our present uncertainty respecting them can have on the subject in question, I have employed three different estimates of the oblateness $\frac{1}{330}$, $\frac{1}{320}$, and $\frac{1}{310}$. With respect to the radius of the equator, that is ascertained with sufficient precision by the mean of the arc extending from Greenwich to Formentera, corresponding to latitude 45° 4′ 18″. The value of the degree in toises is 57010,5, and it is highly probable that in this estimate the error does not amount to so much as half a toise, as it is deduced from an entire arc of 12° 48′ between the two extremities, the latitudes of which have been determined with extreme care, and by a great number of observations.

The following are the logarithms of radius at the equator, which I have employed as adapted to each degree of oblateness,

and opposite to them are placed the corresponding computed estimate of the entire arc between Clifton and Dunnose.

$\frac{1}{330}$ .... 6,5147,400 ..... 2° 50′ 21,972
$\frac{1}{320}$ .... 6,5147,485 ..... 2° 50′ 21,974
$\frac{1}{310}$ .... 6,5147,670 ..... 2° 50′ 21,592

so that the greatest difference is but 0″,38. Let us suppose it 0″,4, or even 0″,5, for the second calculation was made only by means of the western series of triangles, and the third only with the eastern; but even then the error arising from uncertainty in the elements is not half the difference we find between the results of computation and of observations of the fixed stars. It appears therefore, that these elements are by no means to be neglected as a method of verification; and in fact the quantity of 1″,38 is so small, that it is extremely difficult to ascertain this quantity with the very best instruments. Of this we shall find further proof hereafter; but as this discussion is not without its use, I shall enter into some details on this subject.

The measurement in Lapland was performed by means of a double metre, and with a repeating circle of BORDA, sent by the National Institute of France. In order to see to what degree of accuracy the arc computed would agree with that obtained by observations of the pole star above and below the the pole, I assumed an oblateness of $\frac{1}{320}$, and as logarithm of radius I had 6,5147500 expressed in toises and in round numbers. With these elements, and with the data to be found in the work of M. SVANBERG, we have by the western series of triangles 5840″,196 and 5840″,138 by the eastern. So that the mean calculated arc is 1° 37′ 20″,167, while the arc observed was 1° 37′ 19″,566. The difference then is 0″,6 for

the total arc, and 0",37 for the mean degree, or 5,86 toises excess in the linear extent. One can never depend upon quantities so small as this, so that the agreement between the results of computation and actual observation, proves not only the skill of the observers and the accuracy of which their instruments admit; but also that the elliptic elements employed in the calculation are a sufficiently near approximation to the truth to be deserving of confidence.

In the 8th volume of the Asiatic Researches, published by the Society at Calcutta, are contained the details of another measurement performed in 1802, by Major WILLIAM LAMBTON in Bengal, on the Coromandel coast. In this undertaking, which was executed with great skill and attention, Major LAMBTON employed Bengal lights as signals, chains for the linear measures, and a theodolite, and a zenith-sector made by RAMSDEN. The base measured was 6667,740 fathoms reduced to the level of the sea, and to the temperature of 62° FAHRENHEIT; and the stations were so chosen, that four of the sides of the triangles were almost in the same line, and nearly parallel to the meridian at the southern extremity of the arc, so that their sum but little exceeds its whole extent. The lengths of these arcs in fathoms reduced to the meridian are thus given in the Memoir of Major LAMBTON.

AB 20758,13 north latitude of A 11° 44′ 52″,59
BC 17481,245
CD 22237,04 north latitude of E 13° 19′ 49″,018
DE 35246,43

From these data Major LAMBTON deduces the degree of the meridian to be 60435 fathoms, or 56762,3 toises. By applying to this the same elements as we did to the measurement

by SVANBERG, we have the entire arc measured equal to 1° 34′ 55″,896; so that the difference between the results of calculation and of the observations, is only 0″,532 for the whole arc, or 0″,337 for the mean degree. The elliptic hypothesis and observation agree more correctly in this instance, for the difference is rather less than in that of Lapland, although the two arcs are very nearly of the same extent. Thus the degree on the meridian measured in Bengal, in the latitude of 12° 32′ 21″ north, cannot be supposed to exceed Major LAMBTON's estimate by more than 5,22 toises; and it is extremely difficult to speak with certainty to quantities so small as this.

The same observer also measured one degree perpendicular to the meridian, by means of a large side of one of his triangles cutting the meridian nearly at right angles, and of which he observed the azimuth at the two extremities. The data from which his results may be verified are these:

Length of the chord of the long side in English feet AB = 291197,20.

Azimuth of the eastern extremity A equal to 87° 0′ 7″,54 NW.

Azimuth of the western extremity B equal to 267° 10′ 44″,07 NW.

North latitude of A 12° 32′ 12″,27
North latitude of B 12° 34′ 38″,86.

With these data in the triangle formed by the long side, the meridian at B, and the perpendicular from B on the meridian at A, we have the chord of this last arc equal to 290845,8 feet, and the arc itself 290848,03 feet. By applying the method of M. DELAMBRE, we find the azimuth of the extremity B less by 2″ than it was observed to be; so that we have no

reason to suppose a greater error than one second in the observation of each azimuth, and it seems next to impossible to arrive at greater exactness.

The difference of longitude between the points A and B is 48′ 57″,36. With this angle and the co-latitude at A, we have in the spherical triangle right angled at the point A, the extent of the normal arc equal to 2867,330 seconds, and dividing its length in feet by this number, we have for the degree perpendicular to the meridian, at the extremity A, 60861,20 fathoms, or 57106,5 toises. Now these values are precisely what we find on the elliptic hypothesis, with an oblateness of $\frac{1}{320}$ or $\frac{1}{310}$; and in short, the correspondence between the hypothesis and the measures of Major LAMBTON, is as complete as can be wished. Major LAMBTON, indeed, finds the degree on the perpendicular too great by 200 fathoms, but this arises from a mistake in his calculation.

Lastly, I shall apply the same method, and see how nearly the elliptic hypothesis agrees with the last measures taken in France, which merit the highest degree of confidence both with respect to the observers who have executed it, and the means which they had it in their power to employ. I have taken only the arc between Dunkirk and the Pantheon at Paris, from the data published by the Chevalier DELAMBRE in the 3d Vol. of the Measurement of the Meridian. I employed the same elements and similar calculations to those made on the English arc. The oblateness of $\frac{1}{330}$ gives the difference between the parallels equal to 7883,615 seconds by the eastern series of triangles, and 7883,617 by the western series. The mean of these 7883,616 may be taken as the true extent of the total arc.

The two other elements give for this quantity 7883″,621

and 7883″,493, or 2° 11′ 23″,6 and 23″,49, as the calculated extent of the arc. But the arc observed was 2° 11′ 19″,83, according to M. DELAMBRE, and 2° 11′ 20″,85 according to M. MECHAIN; so that the least difference between the calculation and the observations will be 2″,64. M. DELAMBRE is of opinion, that the latitude of Dunkirk, which is supposed to be 51° 2′ 9″,20, should be diminished; and in fact the distance between the parallels of Dunkirk and Greenwich, which is 25241,9 toises, gives by the mean of the three assumed ellipticities 26′ 32″,3 for the difference of latitude. After deducting this quantity from 51° 28′ 40″, the supposed latitude of Greenwich, there remains 51° 2′ 7″,7 or 8″, for that of the tower at Dunkirk. If from this again we deduct the calculated arc 2° 11′ 23″,5, we have 48° 50′ 44″,5 for the latitude of the Pantheon, while, according to the observations of M. DELAMBRE, it is 49″,37, or 48″,35 by those of M. MECHAIN. If various circumstances, with regard to unfavourable weather, and also others of a different kind connected with the revolution, and of which M. DELAMBRE complains with much reason, have occasioned some uncertainty with respect to the observations at Dunkirk, still the numerous observations made at Paris, both by him and by M. MECHAIN at a more favourable season, and in times of perfect tranquillity, render the supposition of an error of 4 seconds in the latitude of the Pantheon wholly inadmissible. It is, however, too true, that such errors are possible, and it is only by careful perseverance, and by repeated verification, that they are to be discovered and removed, as we have seen to be highly probable with respect to the station at Arbury Hill.

But the same celebrated observer, M. MECHAIN, who handled

instruments with great delicacy, and was possessed of peculiar talents for this species of observation, has given us an instance of singular irregularity in the observations made at Montjui and at Barcelona.

The latitude of Montjui, determined by a very long and regular series of zenith distances, is full 3″,24 less than that deduced from a similar series of observations made at Barcelona, with the very same instruments, and with equal care. Moreover, there is reason to think, from other observations, that the latitude of Barcelona (which is supposed to be 45″) ought to be diminished still one second, so that the difference between the observations at Montjui and at Barcelona will probably amount to as much as 4″. Local attractions are supposed to have been the cause of this irregularity; but then the latitude, as deduced from observations made at Barcelona, should have been less than it appeared by those made at Montjui itself; for the deviation of the plumb-line (or of the spirit contained in a level) *could only* be occasioned by the little chain of land elevated to 120 or 130 toises, which passes to the north of Barcelona in a north-easterly direction. Now since the deviations arising from this source would be northward, the zenith distance of circumpolar stars would be augmented by that deviation, and consequently the latitude deduced therefrom would be diminished by just so much. But here the contrary occurs; for the latitude of Montjui deduced from the observations at Barcelona is 48″,23, whilst that obtained by direct observations at Montjui is only 45″. Hence it seems probable, that the cause of this irregularity must be sought elsewhere, and that it is not likely to be discovered without repeating over again the same observations.

Moreover it does not follow that the latitudes of two places are correct, because the declinations of the stars deduced from them correspond; for the deviations caused by local attractions, or from any other source, are made to disappear in correcting the declination, but remain uncorrected in the latitude of each.

Lieut.Col. MUDGE is also of opinion, that the irregularity in the value of his degree may be ascribed to deviation of the plumb-line, occasioned by local attractions. This is certainly very possible, and may be decided by an examination of all circumstances on the spot. But if there be really an error of 1″ in the extent of the whole arc, this should rather be ascribed to some defect in the observations themselves, than to any extraneous source; for the observations of different stars give results that differ more than 4 seconds from each other.

I shall now conclude this Memoir, by expressing a wish, which men of science in England have it more in their power than any others to gratify; I mean by making new measurements in the southern hemisphere. Those which have been made hitherto in the northern hemisphere are extremely satisfactory by their agreement, and give us great reason to presume that the general level of the earth's surface is elliptical, and very regularly so; and hence we might expect the opposite hemisphere to be equally so, and to be a portion of the same curve. Nevertheless the degree measured at the Cape of Good Hope by LACAILLE, in latitude 33° 18′ appears to indicate an ellipse of less eccentricity, or of greater axis; for the linear extent of 57037 toises, corresponds to the measure of a degree in latitude 47° 47′ in the northern hemisphere. If now we calculate the arc as before, with an oblateness of

$\frac{1}{320}$, and with the sides of LACAILLE's triangles reduced to the meridian, we find it greater by 10″ than it was found to be by observations of the stars. An error of 10 seconds, by an astronomer so skilful and scrupulous as LACAILLE, is too extraordinary to be admitted as probable. It is true, that there was a greater error well ascertained to have occurred in the measurement in Lapland, amounting to 13 seconds; but the academicians engaged in this undertaking were by no means equally conversant with observations as LACAILLE.

There remains therefore but one method of removing all doubt on this subject, and this is to repeat and verify the measurement at the Cape, and, if possible, to extend it still farther to the north. The same Major LAMBTON, who has succeeded so well in Asia, and is in possession of such perfect instruments for the purpose, would be singularly qualified for a similar undertaking in Africa, and would furnish us with a measurement in the other hemisphere, as much to be relied upon as the former. He would have the glory of deciding two important questions by his own observations; first, the similarity and magnitude of the two hemispheres; and, secondly, the degree of reliance to be placed on the elliptic hypothesis.

It might be still further desirable, if other measurements could also be undertaken, either in New Holland, or in Brazil; for though neither of these countries differs much in latitude from the Cape of Good Hope, they are so remote in longitude, that a correspondence of measures so taken would nearly establish the similarity of all meridians.

Note.

I shall now explain the formulæ employed in deducing the results to which I have come in the foregoing Memoir. The demonstration of them is to be found in the work of M. DELAMBRE, on the Meridian.

In the first place, let a be the radius of the equator, e the eccentricity, ψ the latitude of one extremity of a side, or arc, in any series of triangles, and θ the azimuth of that side. The radius of curvature of this arc will be expressed by

$$\frac{1}{R1} = \frac{\left(1 + \frac{e^2}{1-e^2} \cdot \cos.^2\psi \cdot \cos.^2\theta\right)}{R} \text{ and } \frac{1}{R} = \frac{(1-e^2 \cdot \sin.^2\psi)^{\frac{1}{2}}}{a}.$$

Hence we see that R is the radius of the arc at right angles to the meridian. One may in general neglect the azimuth, and take the last radius for the radius R1. Now, in computing the arc between Clifton and Dunnose, I have supposed the oblateness to be $\frac{1}{330}$ or $e^2 = \frac{669}{330^2}$, and log. $a = 6{,}5147200$ expressed in toises.

The latitude of the southern extremity of the base is the same as that of Clifton, and its azimuth, if we choose to attend to it, is nearly 335° 23′. This base, considered as an arc of a circle, is reduced to its sine by the formula $\varepsilon = \log. \varepsilon - \frac{K \cdot \varepsilon^2}{6R^2}$, (K being the modules of the table of logarithms, so that log. K = 9,6377843.)

By means of the logarithmic sine of the base, and the angles

of the triangles, considered as spherical, the logarithmic sines of the sides in the series were next computed, and then reduced to logarithms of the arcs themselves by the formula $\log. \varepsilon = \log. \sin. \varepsilon + \frac{K . \sin.^2 \varepsilon}{6R^2}$.

For the purpose of making this last reduction, it is sufficient to take a single value of R, corresponding to the mean latitude of the entire arc 52° 2′ 20″. It was thus that the table was formed of logarithmic sides considered as arcs.

Let m be one of these arcs, and let us represent by $\delta\psi$ and $\delta\psi''$ its value reduced to the meridian, the one in toises, the other in seconds of a degree, and we shall have the following formulæ;

$\delta\psi = m . \cos. \theta - \left(\frac{m^2 . \sin.^2 \theta}{2R}\right) . \operatorname{tang}. \psi - \left(\frac{m^2 . \sin.^2 \theta}{2R}\right) . \left(\frac{m . \cos. \theta}{3R}\right) . (1 + 3 . \tan.^2 \psi)$

$\delta\psi'' = \left(\frac{\delta\psi}{R . \sin. 1''}\right) + \left(\frac{\delta\psi}{R . \sin. 1'}\right) . e^2 . (1 + e^2) . \cos.^2 \psi . \left\{1 \mp \left(\frac{3 \tan. \psi}{2}\right) . \left(\frac{\delta\psi}{R}\right)\right\}$: the superior sign being taken when the latitude ψ'' is greater than ψ, and the inferior when it is less.

The correction dependent on the convergence of the meridian for the azimuths is $\delta\theta = \left(\frac{m . \sin. \theta}{R1 . \sin. 1''}\right) . \left(\frac{\sin.^2 (\psi + \psi')}{\cos. \psi' . \cos. \frac{1}{2} \delta\psi''}\right)$.

Hence the azimuth of the first station seen from the second and reckoned westward from the north, is $\theta' = 180° + \theta + \delta\theta$.

If P″ be put for the difference of longitude between two points distant by an arc which measures m, we have $\sin. P'' = \frac{\sin. m . \sin. \theta}{\cos. \psi}$, $\log. \sin. m = \log. \left(\frac{m}{R1}\right) - \frac{K}{6} . \left(\frac{m}{R1}\right)^2$, and $\log. P'' = \log. \left(\frac{\sin. P''}{\sin. 1''}\right) + \frac{6}{K} . (\sin. P'')$.

The arc of the meridian, between Greenwich and Formentera, is so fortunately situated, that its middle point is in latitude 45°. Its whole extent measures 12° 48′ 44″, and the distance between the parallels, in linear measure, was found to be 730430,7 toises. Hence the mean degree, corresponding to the latitude of 45° 4′ 18″, is 57010,5 toises; and if we multiply this number by 90°, we get one-fourth part of the meridian of the earth.

The correction to be deduced for oblateness is 58, 59, or 61 toises, according as it is assumed to be $\frac{1}{330}$, $\frac{1}{320}$, or $\frac{1}{310}$, and if we take the mean of these, we have the fourth part of the meridian Q = 5130886 toises; and hence the metre = 44330867 lines; so that the value of the metre turns out to be almost entirely independent of the elliptical form of the earth.

The radius of the equator is derived from the expression $\log. a = \log. \left(\frac{2Q}{\pi}\right) + K . \left(\frac{1}{2} . \varepsilon + \frac{1}{16} . \varepsilon^2 - \frac{1}{48} . \varepsilon^3\right)$, ε being the oblateness, and π the periphery of a circle = 3,1416.

In order to compare any degrees measured with those obtained on the elliptic hypothesis, we have a very simple formula. Let m and m' be the values of two degrees on the meridian, of which the mean latitudes are $\psi 1$ and $\psi 2$; in comparing the analytic expressions for these two degrees, developing them, and then making $\psi = 45°$, we have $m' = m . (1 - \frac{1}{2} . p . \cos. 2\psi 2 + g . \cos.^2 2\psi 2)$, $m = 57010{,}5$ toises, $p = \frac{3}{2} e^2 . (1 + \frac{1}{2} e^2) . \frac{\sin. 1°}{1° . \sin. 1''}$, and $g = \frac{15}{64} . e^4 . \left(\frac{\sin. 2°}{1° . \sin. 1''}\right)$.

And then we shall find that the oblateness $\frac{1}{320}$ gives 57075,66 and 57192,38 toises for the degrees in England and Lapland.

I shall here subjoin one reflection more, which appears of

importance. The oblateness of the earth is a quantity which varies considerably, by the least difference in the elements on which it depends. Accordingly it is not surprising, that its value fluctuates between two proportions which differ sensibly from each other. To illustrate this, let p be the function which serves to determine the oblateness of the earth, so that $\frac{1}{\epsilon} = p$. When this equation varies $-\delta\epsilon = \epsilon^2 . \delta p$.

Now the coefficient ϵ^2 being very great, we see why the least variation in the elements of the function p, occasions so considerable a variation in the denominator of the oblateness. This is precisely what happens in the lunar equations dependent on the figure of the earth, and which M. LAPLACE has deduced from his beautiful theory. Thus, for example, in the inequality that depends on the longitude of the moon's node, which he has determined analytically with so much precision, the numerical coefficient found by BURG gives $\frac{1}{305}$ for the oblateness; but if this coefficient be diminished by $0'',665$, then the oblateness becomes $\frac{1}{320}$, so that a variation even to this small amount in the coefficient augments the denominator of the oblateness nearly $\frac{1}{20}$ part.

The same happens with regard to the pendulum vibrating seconds; for, supposing its length at 45° to have been correctly ascertained by M M. BIOT and MATHIEU, if we wish to know the length of a second's pendulum at the equator, corresponding to an oblateness of $\frac{1}{310}$, we find it to be 439,1810 lines. Now this length differs from that determined by BOUGUER only by 0,029 of a line, and M. LAPLACE even thinks that the result of BOUGUER should be diminished by about double this quantity. We see from hence how much these little differences, whether produced by errors of observation,

or irregularities in the earth itself, are liable to affect the denominator of the fraction expressing the oblateness.

Fortunately, it seems probable, that the utmost latitude of our present uncertainty is between the limits of 330 and 310, and the mean of these may be considered as a very near approximation to the truth.

XVIII. *An Account of some Experiments on different Combinations of Fluoric Acid. By* John Davy, *Esq. Communicated by Sir* Humphry Davy, *Knt. LL.D. Sec. R. S.*

Read June 11, 1812.

Introduction.

Two years ago, I engaged, at the request of my brother, Sir H. Davy, in an inquiry respecting the nature of common fluoric acid gas. My principal object was to ascertain whether silex is essential to its constitution, and whether the proportion is constantly the same. This subject, and experiments on the fluoric and fluoboracic acids, occupied me for about six months. Since that time, the work of M. M. Gay Lussac and Thenard has appeared, entitled "Recherches Physico-Chemiques," in the second volume of which is an elaborate dissertation on fluoric acid. These philosophers, I find, have anticipated many of my results, and consequently very much abridged my labour of detail in the following pages. To repeat what is already known would be useless, I shall therefore confine myself to describe what I have observed, which appears to me yet novel, or different from the observations of the French chemists. The order which I shall pursue, will be that which I observed in my experiments. I shall divide what I have to advance into four parts. The first part will relate to the silicated fluoric acid gas, and to the subsilicated fluoric acid; the second to the

combinations of these acids, and of pure fluoric acid with ammonia; the third to fluoboracic acid; and the fourth to its ammoniacal salts.

SECT. 1. *On silicated fluoric acid Gas, and subsilicated fluoric Acid.*

The facts which have already been published by M. M. GAY LUSSAC and THENARD and others, appear to me to be sufficient to prove that pure fluoric acid has not yet been obtained in the gaseous state, and that silex, or boracic acid, is requisite that it may assume this form. Were more evidences necessary, I could advance many in point. One circumstance only I shall mention, proving that common fluoric acid gas is perfectly saturated with silex. I have preserved this gas, made by heating, in a glass retort, a mixture of fluor spar and sulphuric acid, for several weeks over mercury in a glass receiver uncoated with wax, without observing the slightest erosion to be produced.*

This gas, with great propriety, has lately been called silicated fluoric. Before I proceed to its analysis, I shall notice what method I have found the best for obtaining it. I have, for a considerable time, long before M. M. GAY LUSSAC and THENARD's work was published, added to the mixture of fluor spar and sulphuric acid, a quantity of finely pounded glass, and have thus procured the gas with the greatest facility. The advantages of this addition are considerable. The retort is saved, which otherwise, in less than one operation, would be

* The sides of the receiver indeed became obscure; but this was not from erosion, but from deposition, as appeared from the transparency and polish of the glass being readily restored by slight friction. What the deposition was, I am ignorant of. After several weeks it was so trifling, as to give only a slight degree of opacity to the receiver.

destroyed; and a much larger quantity of gas is procured from the same materials, and with less trouble and less heat; the action indeed at first is so powerful, that gas begins to come over before the application of heat is made, and a very gentle one only is required to continue its production.

Previous to its analysis, it was necessary to ascertain the specific gravity of the gas. This I have endeavoured to do. The gas, the subject of experiment, was quite pure, being totally condensed by water. A Florence flask was exhausted; in this state, weighed by a very delicate balance, it was

	= 1452.2 grains.
Filled with common air -	= 1452.2 + 10.2
Again exhausted - -	= 1452.2
Filled with silicated fluoric gas	= 1452.2 + 36.45

Hence as 10.2 : 31 :: 36.45 :: 110.78.

Thus it appears, that 100 cubic inches of silicated fluoric acid gas, at ordinary temperature and pressure, are equal to 110.78 grains.

When silicated fluoric acid gas is condensed by water, it is well known that part only of the silex is deposited. To obtain the whole, in order to ascertain the proportion in the gas, I have employed ammonia in excess. 40 cubic inches of the gas (barom. 30, therm. 60) were transferred in portions of 10 cubic inches, at a time to a solution of ammonia. The silex precipitated was carefully collected on a filter, and washed till the water that passed through it, ceased to be affected by nitrat of lime. It was next dried, and strongly heated in a platina crucible. It weighed 27.2 grains, and was pure silex. Supposing fluoric acid to be the remaining 17.1 grains, which added to 27.2 grains are equivalent to the weight of 40 cubic

inches of the gas, it appears that 100 parts by weight of this gas consist of

61.4	silex
38.6	fluoric acid
100.0	

That this estimate may be correct, it is evident, that ammonia should have the property of precipitating the whole of the silex of silicated fluoric gas; which I shall not now endeavour to prove, but leave it to be considered in another part of the paper.

There is no improbability attached to the idea, that silicated fluoric acid gas may, from the manner in which it is prepared, contain a proportion of alkali. To discover whether this was the case, a solution of nitrat of lime was added to the ammoniacal solution neutralized by nitric acid, from which the silex in the preceding experiment had been removed. The precipitate of fluat of lime was separated by filtration. The filtered liquid was evaporated to dryness; and the ammoniacal salt heated in a platina crucible till it was entirely dissipated. The residue had the appearance and taste of quick lime. It was dissolved in acetic acid, and the solution yielded sulphat of lime on the addition of sulphat of ammonia. The liquid was evaporated to dryness, and when the residuum had been heated to dull redness, nothing remained but a little white powder, weighing about a grain, and having all the properties of gypsum. Thus it appears that silicated fluoric acid gas contains no alkali.

My next object was to ascertain the composition of common liquid fluoric acid—that acid obtained by the decomposition of

silicated fluoric acid gas by water, and which, on account of the separation that occurs of part of the silex, may, with greater propriety, be called subsilicated fluoric acid. For this purpose, 43.21 cubic inches, barom. 30.4, therm. 50, or 44 cubic inches at common temperature and pressure, were successively added, two cubic inches at a time, to one cubic inch of distilled water in a small jar over mercury. The whole of this, the gas being pure, was readily condensed. The temperature was somewhat raised. The silex precipitated, formed a gelatinous mass of a blueish colour, which had absorbed all the water like a sponge, so that none appeared fluid. This gelatinous mass was carefully transferred to a filter, and washed with distilled water till it was rendered insipid and incapable of reddening litmus paper. It retained its blueish hue only whilst moist. When dried and ignited, it was in thin lamellæ, and of a snow-white colour, and surprisingly bulky. It weighed 7.33 grains, and was found to be pure silex. Thus it appears that the subsilicated fluoric acid formed by the decomposition of 44 cubic inches of silicated fluoric acid gas contains 7.33 grains of silex less than the gas itself. Consequently independent of water, which no doubt is essential to this acid, 100 parts of it seem to consist of

54.56 silex
45.44 acid

100.00

I have endeavoured to ascertain what quantity of silicated fluoric acid gas a given quantity of water will condense. In one instance $\frac{19}{100}$ of a cubic inch of distilled water absorbed 51 cubic inches, barom. 30.5, therm. 60. The gas was added to

the water in a jar over mercury, as fast as it was absorbed. The experiment was stopped, when the gas, after having remained in contact with the water a whole night, ceased to be diminished. According to this result, the proper correction being made for the additional pressure, water decomposes about 263 times its bulk of silicated fluoric acid gas.

Dr. PRIESTLEY observed, that muriatic acid gas reproduced silicated fluoric gas from the crust of silex formed, when the latter is condensed by water.* This experiment I have repeated, and as it appears to show more correctly the quantity of gas water can condense, I shall describe the result. 2.4 cubic inches of muriatic gas were added to a drop of water, that had previously absorbed one cubic inch of silicated fluoric gas, in a jar over mercury. There was an immediate absorption equal to $\frac{2}{10}$ of a cubic inch. The mixture of silex and subsilicated fluoric acid effervesced, and from an apparent solid became fluid, the whole of the silex gradually disappearing. After the first mentioned absorption, there was no farther. The gas produced was silicated, as appeared from the crust it deposited when removed to water, and the liquid formed was pure muriatic acid, for decomposed by concentrated sulphuric, it afforded merely muriatic acid gas, without any silicated fluoric. The evident conclusion from the preceding result is, that water condenses equal quantities of the muriatic and silicated fluoric acid gasses, and consequently that the first estimate is too low, and instead of 263 times its bulk, it is probably more correct to say that water to be saturated requires at least 365 times its volume. Neither will this estimate appear inconsistent with the former result, when the

* Vide PRIESTLEY on Air, Vol. II. p. 202.

deposition of silex is considered as an obstacle to the free exposure of the surface of the water to the gas.

Subsilicated fluoric acid is decomposed by ammonia and the fixed alkalies, and by all the earths that I have made trial of. It is also decomposed by the sulphuric acid and the boracic, as well as by the muriatic acid gas.

Of the particular changes which occur when it is acted upon by the alkalies, I defer giving any account at present, as it is my intention to do it in the next section.

To learn the effect of heat on it, a small quantity of strong acid, pure and transparent was introduced into a retort connected with mercury. A spirit lamp being applied about three cubic inches of silicated fluoric acid gas were produced. The neck of the retort was lined with silex in a gelatinous state, and much liquid subsilicated fluoric acid, that had distilled over, was condensed in the colder part of the neck, and was absorbed by bibulous paper previously introduced, to prevent the distilled fluid from entering the jar for the reception of the gas. When the whole of the acid in the bulb of the retort had been evaporated, little or no silex remained.

The general result of this experiment is very different from that which Dr. PRIESTLEY, who first made it, obtained. Instead of silicated fluoric acid gas, he procured " vitriolic acid air," sulphureous acid gas.

I have tried also the effect of heat on the silicious crust, formed by the decomposition of silicated fluoric acid gas, by water; but could obtain no sulphureous acid gas, as Dr. PRIESTLEY did only a small quantity of silicated fluoric.

The correctness of Dr. PRIESTLEY's observations cannot be doubted. I can only account for his results, by supposing that

some sulphuric acid in consequence of the high temperature employed in making the gas was volatilized, and mixed with the subsilicated fluoric acid, and that mercury also was present from the acid being prepared over this metal.

These experiments too oppose another statement relative to a method prescribed for making fluoric acid gas free from silex, by merely heating strong subsilicated fluoric acid in a retort, and collecting the gas over mercury. It is asserted, in chemical works of some reputation, that this process is successful. I have never found it so, having always obtained results similar to those above stated. This, I suppose, is one of the many errors that have secretly crept into repute, and has been believed, because never subjected to the test of experiment.

The action of concentrated sulphuric acid on subsilicated fluoric acid, is similar to that of muriatic acid gas, occasioning a disengagement of silicated fluoric acid gas. Facts which appear to prove, that water is absolutely essential to the existence of this acid.

Boracic acid decomposes it, in a very different way, not from any predominant affinity for the water, but in consequence of a stronger attraction for the fluoric acid itself. Silicated fluoric acid of course is not produced; but liquid fluoboracic acid and the silex is precipitated in a gelatinous state, as when ammonia is employed.

These are the principal facts I have to notice respecting this acid. Before I conclude, I shall briefly mention a few other circumstances. Applied to the tongue, in its concentrated state, it produces a very painful sensation, like that which strong muriatic acid does, and it has a very similar effect on the cuticle. It does not appear to erode glass, for I have kept it in

bottles of this substance more than a month without any action being perceptible. Exposed to the air, it slowly and almost completely evaporates, there being only a very trifling silicious residue; and when gently heated in an open vessel, it is rapidly dissipated in white fumes.

SECT. II. *On the Combinations of silicated fluoric acid Gas, and the subsilicated Fluoric, and the fluoric Acids with Ammonia.*

M. GAY LUSSAC has shewn that silicated fluoric acid gas, like carbonic acid gas, condenses twice its volume of the volatile alkali.* The experiment I have several times repeated, and constantly with the same result, no difference appearing when the acid gas was added in great excess to the alkaline, or the alkaline to the acid. This being the case, and knowing the specific gravities of the two gasses,† 100 parts by weight of silicated fluat of ammonia seem to consist of

24.5 ammonia
75.5 acid
―――
100.0

Silicated fluat of ammonia volatilizes unaltered, if heated by a spirit-lamp in the vessel in which it is formed, and provided moisture be entirely excluded.

Like silicated fluoric acid gas itself, this salt is decomposed by water, and a similar precipitation of silex occurs, and in the same proportion. Thus the salt formed by the union of 30

* Vide Mém. d'Arcueil, Tom. II.

† According to Sir H. DAVY, 100 cubic inches of ammonia, barom. 30, therm. 60, weigh 18 grains. It is this estimate which I have taken.

cubic inches of silicated fluoric gas, and 60 of volatile alkali (barom. 30, therm. 60) in a small glass jar over mercury, being carefully collected and introduced into water, afforded five grains of pure silex, weighed after being well washed and heated to redness.

The saline solution, since part of the silex of the silicated fluoric acid gas is separated during its production, appears to be a subsilicated fluat, or a combination of subsilicated fluoric acid and ammonia. Another mode of making it, more directly proves that this is its composition. When ammonia is added to the subsilicated fluoric acid in excess, this salt is formed without any precipitation. From these facts, it may be concluded, that independent of water, which appears to be essential to its existence, 100 parts of it consist of

28.34 ammonia
71.66 acid

100.00

Subsilicated fluat of ammonia has a pungent saline taste. It just perceptibly reddens litmus paper. Slowly evaporated, it forms small transparent and brilliant crystals. The largest I could obtain, appeared to be tetrahedral prisms. The solid salt is very soluble in water; but is not deliquescent. When heated it appears to sublime unaltered. It is curious that the solution of this salt, when evaporated by a heat near its boiling point, powerfully erodes the glass or porcelain vessel, and a residuum of silex appears, on the addition of water, to redissolve the salt. This erosion and residue of silex I have seen produced three times following, with the same quantity of salt. I mention the fact, which, I believe, was before observed by SCHEELE, without

attempting an explanation of it. It may perhaps be said, that as the water evaporates, the affinity of the subsilicated fluat for silex increases.

Subsilicated fluat of ammonia is decomposed by the sulphuric acid, and by muriatic acid gas, and also by the fixed alkalies and by ammonia.

Sulphuric acid expels from it, silicated fluoric gas and hydrated fluoric acid fumes.

Muriatic acid gas acts slowly on it, and effects its decomposition apparently through the medium of its water. A little of the crystalline salt was introduced into muriatic acid gas in a jar over mercury. In a short time some silicated gas was produced, as the silicious deposition, on the addition of water, indicated. Strong muriatic acid was substituted for the acid gas. Now no apparent change took place, for on evaporating the acid, the residue, decomposed by sulphuric acid, afforded only silicated fluoric acid gas.

The alkalies form by the decomposition of this salt, the same compounds that they do by their action on subsilicated fluoric acid.

Potash expels the ammonia, and produces the silicated fluat and fluat of potash, as M. M. GAY LUSSAC and THENARD have described.

The changes occasioned by soda appeared to me similar; but the gentlemen just mentioned, assert that this alkali precipitates the whole of the silex, and does not form a triple salt with it and part of the acid.

Ammonia seems to me to separate completely the silex, and by uniting with the pure acid to constitute a true fluat. MM. GAY LUSSAC and THENARD are of a different opinion. They

say that the whole of the silex cannot by this method be removed, but only the principal part. Their reason for this belief, is, that on repeatedly evaporating the salt after the addition of ammonia and redissolving it, they have each time observed a residue of silex. If they employed metallic evaporating vessels, the results of my experiments do not agree with theirs; for making use of platina for this purpose, and adding an excess of ammonia, I never detected traces of silex on evaporating the filtered fluat. But our results agree, if they employed glass or porcelain vessels, which fluat of ammonia has the property of corroding.

I now proceed to the consideration of fluat of ammonia; but before I describe some of the properties of this fluat which I have observed, I shall briefly mention the means pursued for ascertaining the proportions of its constituent parts.

The composition of subsilicated fluat of ammonia being known, that of the fluat (granting what is already advanced respecting its formation to be correct) may be inferred from the proportion of silex, that a given quantity of ammonia will precipitate. 18 cubic inches of ammoniacal gas were condensed by $\frac{1}{4}$ of a cubic inch of distilled water in a small glass tube over clean mercury. This ammoniacal solution was added to a clear filtered solution of subsilicated fluat of ammonia. A precipitate of silex was immediately produced. After several hours standing, this precipitate was collected on a filter, well washed and dried and heated to redness. It was pure silex, and weighed 1.6 grains. This experiment, like all the preceding, was repeated, and the result confirmed. In both instances there was an excess of subsilicated fluat. The precipitations were made in a platina vessel, and the solutions were neither heated before or after

the separation of the silex. Calculating from this result, 100 parts of fluat of ammonia seem to consist of

76.4	ammonia
23.6	fluoric acid
100.0	

Water appears to be a constituent part of this salt.

It may be rendered neutral by means of a gentle heat, which expels the excess of ammonia employed in its formation. In its neutral state, it has a strong saline taste, and it readily deliquesces when exposed to the atmosphere. Like the neutral carbonats, it is decomposed by heat; but there is this difference between them, part of the pure alkali is expelled instead of the acid, and an acid fluat of ammonia is formed. A gentle heat only is required for the purpose, that of boiling water is nearly sufficient. When the heat is much stronger, the salt fuses and passes off in dense fumes of a most peculiar suffocating odour. The effects of these fumes, when inhaled, are very powerful and disagreeable, and even dangerous, I might venture to say, were I to speak from my own experience. In one instance, when I inhaled only a small quantity, they produced in a few minutes a violent cough and catarrh, and apparent accumulation of blood in the neck and head, and symptoms altogether not unlike those the attendants of apoplexy, which continued for about a quarter of an hour, and then slowly diminished, and gradually disappeared without leaving any permanent bad effect. The fluat of ammonia, when heated in a metallic vessel, appears to sublime unaltered. But the result is different when the experiment is made in a glass one. Ammonia is expelled, the glass is corroded, and subsilicated fluat

of ammonia is formed and sublimed. Its action on glass is so powerful, that I have successfully employed it instead of fluoric acid itself, for etching on this substance. It has one advantage, that it is more manageable. The solution may be applied by means of a hair pencil or a common pen to the glass, and the erosion will be produced by exposure to a moderate temperature.

The fixed alkalies, and all the earths that I have tried, decompose this salt; they expel the ammonia, and form true fluats with the acid itself. I have examined all the fluats thus formed, and have endeavoured to ascertain the proportions of their constituent parts; but I am not sufficiently satisfied of the accuracy of the results, to venture to give an account of them.

SECT. III. *On Fluoboracic Acid Gas.*

MM. GAY LUSSAC and THENARD, who first discovered this gas, obtained it by heating strongly, in an iron tube, a mixture of fluor spar and fused boracic acid. I have found that it may be more easily procured, in greater abundance, and at less expence, by gently heating, in a common glass retort, a mixture of finely pounded boracic acid * and fluor spar with concentrated sulphuric acid. 1 part by weight of fused boracic acid, 2 parts of fluor spar, and about 12 of sulphuric acid appear to be the proportions best adapted for the purpose. This method will require no explanation when it is considered that boracic acid, as has already been observed, precipitates silex from liquid subsilicated fluoric acid. If the heat is gentle, not

* Common calcined borax answers the same end, but no so well. Its only recommendation to preference is cheapness.

nearly sufficient to occasion the ebullition of the sulphuric acid, and the proportions just recommended are used, the retort will not be injured, and pure fluoboracic acid gas will be produced in abundance. When the gas ceases to come over, if the heat is raised, more will be evolved, and there will be distilled over at the same time, a viscid fluid, which is a compound of sulphuric acid and fluoboracic acid gas. Now the operation should be stopped, if the object is to obtain merely pure fluoboracic gas, a long continuation of the heat producing some silicated fluoric. Before quitting the subject, it should be observed that the quantity of sulphuric acid employed is of considerable consequence to the success of the experiment. If too much is used, there is a great loss of gas from the property which sulphuric acid has of absorbing fluoboracic acid gas; and if too little is employed, it soon becomes diluted, and loses the power of generating the gas, though it may still decompose the fluor spar. Both extremes, therefore, are to be avoided, and the proportion of acid mentioned above, as far as my experience goes, appears to be the best.

I have endeavoured to ascertain the specific gravity of fluoboracic gas.

The flask exhausted weighed	1400.5 grains.
Filled with common air -	1400.5 + 6.2
Again exhausted - -	1400.5
Filled with pure acid gas -	1400.5 + 14.7.

Thus it appears that 100 cubic inches of fluoboracic gas are equal to 73.5 grains.

M M. GAY LUSSAC and THENARD have described the compound of this gas and water, a fuming fluid, in many respects similar to concentrated sulphuric acid. Like this acid, I have

observed that it possesses a slight degree of tenacity, so that it has an oily appearance when poured from one vessel to another; and similar in another respect, it possesses the property of charring animal and vegetable substances, and which the French chemists observed belonging to the gas itself. I have found that water condenses more of this, than it does of any other known gas, no less than 700 times its volume. The experiment was then made, barom. 30.5, therm. 50, $\frac{14}{100}$ of a cubic inch of water were introduced into a tube over mercury, and the gas, in portions of 5 cubic inches at a time, was added until 100 cubic inches had been absorbed, when the water was apparently saturated. This acid was of the specific gravity 1.77.

The property which sulphuric acid has of absorbing fluoboracic acid gas has already been noticed. I found that $\frac{1}{2}$ cubic inch of sulphuric acid, of the specific gravity 1.85, condensed 25 cubic inches of the gas, or 50 times its volume. The compound acid was strongly fuming, and appeared more tenacious than pure sulphuric acid, yet not nearly so much so as that compound of the two which distills over during the latter part of the operation of making fluoboracic gas.

This latter compound has some peculiarities. It is so tenacious, that it flows very slowly. It appears to be far more volatile than pure sulphuric acid. When poured into water, a dense white precipitate is formed, the exact nature of which I have not yet satisfactorily ascertained; but which is not produced by the direct compound of sulphuric acid, and the fluoboracic.

SECT. IV. *On the Combinations of fluoboracic acid Gas, and ammoniacal Gas.*

M. GAY LUSSAC has combined fluoboracic acid gas with ammonia. He states, that equal volumes of the two gasses condense each other.* This I have found to be the case, and I have also found that fluoboracic acid gas condenses twice, and even three times its volume of the volatile alkali. The compound observed by M. GAY LUSSAC is solid, white, and opaque, like the ammoniacal salts. The combinations I have obtained are liquid, transparent, and colourless, like water, though they are entirely free from this fluid. They were made by the direct union of the two gasses. 5 cubic inches of ammoniacal gas were added to the same volume of the fluoboracic gas contained in a small jar over dry mercury. There was a complete condensation of both, and the solid salt was the result. 5 cubic inches more of ammonia were introduced. The whole was quickly absorbed, and the solid salt was converted into the transparent fluid. 5 cubic inches more were added, which too were slowly absorbed, but without any change of form.

The solid salt volatilizes in close vessels unaltered, on the application of a gentle heat.

Both fluid compounds, when heated, are rendered solid, from the expulsion of part of the ammonia. Exposure to the air is attended with the same change, and the same effect is produced by the muriatic and carbonic acid gasses.

Knowing the volumes of the acid, and alkaline gasses which

* Vide Mém. d'Arcueil, Tom. II.

combine, it is easy to calculate the proportions of each by weight in the respective salts.

100 Parts consist of	Ammonia.	Acid.
The solid compound	19.64	80.32
The first fluid -	32.9	67.1
The second fluid	42.4	57.6

These combinations are curious in many points of view. They are the first salts that have been observed liquid, at the common temperature of the atmosphere, without containing water. And they are additional facts in support of the doctrine of definite proportions, and of the relation of volumes.

XIX. *On a Periscopic Camera Obscura and Microscope.* By William Hyde Wollaston, *M. D. Sec. R. S.*

Read June 11, 1812.

Although the views, which I originally had of the advantage to be derived from the periscopic construction of spectacles,* naturally suggested to me a corresponding improvement in the *camera obscura*, by substituting a meniscus for the double convex lens, I have hitherto deferred making it known to others, except as a subject of occasional conversation.

Since in vision with spectacles, as in common vision, the pencil of rays received by the eye in each direction is small, the superiority of that form of glass, which disposes all parts of it most nearly at right angles with the visual ray, admits of distinct demonstration; but with respect to the camera obscura, where the portion of lens requisite for sufficient illumination, is of considerable magnitude, although it is evident that some improvement may be made in the distinctness of oblique images on the same principles, yet as the focus of oblique rays is far from being a definite point, the degree in which it may be improved is not a fit subject of mathematical investigation.

I have therefore had recourse to experiments, in order to determine by what construction the field of distinct representation may be most extended; and, I trust, the result will be acceptable to this Society. I shall take the same opportunity

* Phil. Magaz. Vol. XVII. Nicholson's Journal, VII. 143.

to describe an improvement in the construction of the simple microscope, which may also be termed periscopic, as the object of it is to gain an extension of the field of view, upon the same principles as in the preceding instances, namely, by occasioning all pencils to pass as nearly as may be at right angles to the surfaces of the lens. The mode, however, in which this is effected is apparently somewhat different in the practical execution.

In the common *camera obscura*, where the images of distant objects are formed on a plane surface to which the lens is parallel, if the surfaces of the lens be both convex, and equally curved (as in fig. 1); and if the distance of the lens be such, that the images formed in the direction of its axis CF be most distinct, then the images of lateral objects are indistinct in a greater or less degree, accordingly as they are more or less remote from the axis. The causes of this indistinctness may be considered as twofold; for in the first place, all parts of the plane, excepting the central point, are at a greater distance from the centre of the lens than its principal focus; and secondly, the point *f*, to which any pencil of parallel rays passing obliquely through the lens are made to converge, is less distant than the principal focus. On this account, it is in general best to place the lens at a distance somewhat less than that which would give most distinctness to the central images, because in that case a certain moderate extension is given to the field of view, from an adjustment better adapted to lateral objects, without materially impairing the brightness of those in the centre. The want of distinctness, however, is even then only diminished in degree, but is not remedied.

The construction, by which I propose to obviate this defect,

is represented in the second figure, in which are seen the essential parts of a periscopic camera in their due proportion to each other. The lens is a meniscus, with the curvatures of its surfaces about in the proportion of two to one, so placed that its concavity is presented to the objects, and its convexity toward the plane on which the images are formed. The aperture of the lens is four inches, its focus about twenty-two. There is also a circular opening, two inches in diameter, placed at about one-eighth of the focal length of the lens from its concave side, as the means of determining the quantity and direction of rays that are to be transmitted.

The advantage of this construction over the common camera obscura is such, that no one who makes the comparison, can doubt of its superiority; but the causes of this may require some explanation. It has been already observed, that by the common lens, any oblique pencil of rays is brought to a focus at a distance less than that of the principal focus. But in the construction above described, the focal distance of oblique pencils is not merely as great, but is greater than that of a direct pencil. For since the effect of the first surface is to occasion divergence of parallel rays, and thereby to elongate the focus ultimately produced by the second surface, and since the degree of that divergence is increased by obliquity of incidence, the focal length resulting from the combined action of both surfaces will be greater than in the centre, if the incidence on the second surface be not so oblique as to increase the convergence. On this account, the opening E is placed so much nearer to the lens than the centre of its second surface, that oblique rays E*f*, after being refracted at the first surface, are transmitted through the lens nearly in the direction of its

shorter radius; and hence are made to converge to a point so distant that the image (at *f*) falls very nearly in the same plane with that of an object centrally placed.

In the use of spectacles by long-sighted persons, the course of the rays in the opposite direction is so precisely similar, that the same figure might serve to illustrate the advantages of the periscopic construction. For the purpose of seeing the extended page of a book (as at AB) with least fatigue to the eye, that form of lens will be most beneficial, which renders the rays received from each part of its surface parallel; and this is effected by the exact counterpart to the preceding arrangement; for in this case the opening E represents the place of the eye receiving parallel rays from the lens in each direction, instead of transmitting them from a distance towards it.

There is, however, this difference between the two cases, that in the camera obscura a much larger portion of the lens is required to conspire in giving a distinct image of any one object; so that the conformation best adapted for lateral objects, would not be consistent with distinctness at the centre; and hence arises a limit to the application of the principle. On the common construction, the whole lens is so formed as to give brilliancy and distinctness at the centre alone, without regard to lateral objects. In adopting such a deviation from the customary form, as I propose, in favour of a more extended view, some diminution of the aperture is required in order to preserve the desired distinctness at the centre. In my endeavours to ascertain the most eligible form of meniscus for this purpose, I have assumed sixty degrees to be the field of view required. But when so large a field is not wanted,

then a lens that is less curved will be preferable; and the proportion of the radii must be varied according to the angular extent intended to be included.

For the purpose of estimating by what combination of radii any required focal length may be given to a meniscus, I have contrived a diagram by which very much labour of computation may be saved, as a very near result may be obtained by mere inspection. This contrivance is founded on the well known formula for the focal length of any lens $F = \frac{mrR}{R \pm r}$: m being a certain multiple obtained by dividing the sine of refraction by the difference of the sines of incidence and refraction. Hence, in applying this formula to the meniscus, $F : R :: mr : R - r$. In fig. 3, lines expressive of these quantities are so arranged, that by assuming any point F corresponding to the focal length desired, and drawing a line FR through a point R indicating any supposed length of the greater radius, the corresponding length of the other radius will be found where the line drawn intersects the middle line in the diagram.

In laying down these lines, the length and position of AF and AR were assumed at pleasure; and they were divided into any number of equal parts. But the position and length of the middle line Ax was adapted with care to the refractive power of plate glass in the following manner. Since $m = \frac{1}{1{,}505 - 1} = 1{,}98$, a line BC was drawn from the point 10 in the line AR, parallel to AF, and equal to 19,8 divisions of the primary lines; so that if r be $= 10$, then the line BC $= mr$. The distance AC being then divided into ten equal parts, with their subdivisions, afforded the means of continuing the

same scale to any desired length. Since the first line BC was laid down parallel to AF, and equal to mr, any other lines drawn through corresponding numbers 7 and 7, 8 and 8, &c. will be also parallel, and by preserving due proportion, will correctly represent mr. Hence in all positions of the line FR, the same similarity of triangles obtains, and the same proportion of $F : R :: mr : R - r$; and consequently the focal length, corresponding to any assumed radii, is truly ascertained.

For the purpose of duly proportioning the curvatures of flint-glass, a second line Ay might be laid down in a mode similar to the preceding, by adapting the multiple $m = \frac{1}{1,58 - 1} = \frac{19}{11}$ to the different density of this glass.

With respect to the construction of a microscope on periscopic principles, I believe the contrivance to be equally new with the former, and equally advantageous. The great desideratum in employing high magnifiers is sufficiency of light; and it is accordingly expedient to make the aperture of the little lens, as large as is consistent with distinct vision. But if the object to be viewed, is of such magnitude as to appear under an angle of several degrees on each side of the centre, the requisite distinctness cannot be given to the whole surface by a common lens, in consequence of the confusion occasioned by oblique incidence of the lateral rays, excepting by means of a very small aperture, and proportionable diminution of light.

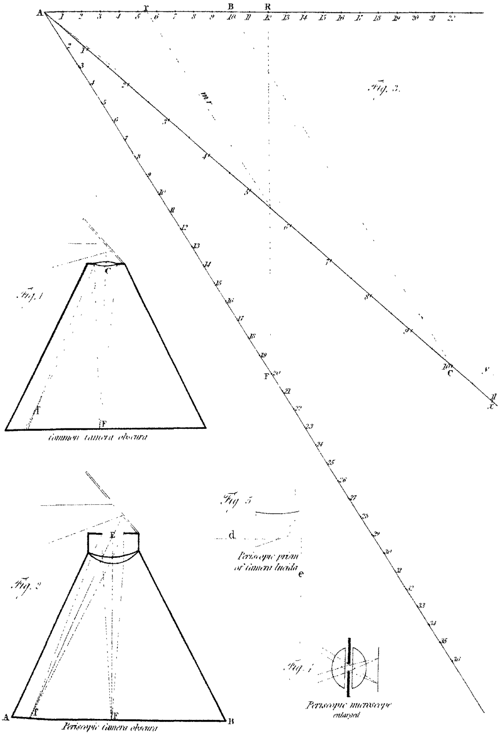
In order to remedy this inconvenience, I conceived that the perforated metal, which limits the aperture of the lens, might be placed with advantage in its centre; and accordingly I procured two plano-convex lenses ground to the same radius,

and applying their plane surfaces on opposite sides of the same aperture in a thin piece of metal (as is represented by a section, fig. 4), I produced the desired effect; having virtually a double convex lens so contrived, that the passage of oblique pencils was at right angles with its surfaces, as well as the central pencil. With a lens so constructed, the perforation that appeared to give the most perfect distinctness was about one-fifth part of the focal length in diameter; and when such an aperture is well centered, the visible field is at least as much as twenty degrees in diameter. It is true, that a portion of light is lost by doubling the number of surfaces; but this is more than compensated by the greater aperture, which, under these circumstances, is compatible with distinct vision.

Beside the foregoing instances of the adaptation of periscopic principles, I should not omit to notice their application to the camera lucida; as there is one variety in its form, that was not noticed in the description which I originally gave of that instrument.*

In drawing, by means of the camera lucida, distant objects are seen by rays twice reflected (*d*, fig. 5), at the same time and in the same direction that rays (*e*) are received from the paper and pencil by the naked eye. The two reflections are effected in the interior of a four-sided glass prism, at two posterior surfaces inclined to each other at an angle of 135 degrees. In the construction formerly described, the two other surfaces of the prism are both plane, through which the rays are simply transmitted at their entrance and exit. But since an eye that is adjusted for seeing the paper and pencil, which are at a short distance, cannot see more distant objects dis-

* Nicholson's Journal, XVII. p. 1. Phil. Magaz. XXVII. p. 343.



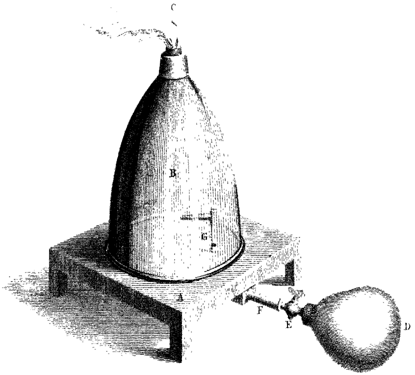
Common camera obscura

Periscopic camera obscura

Periscopic prism or camera lucida

Periscopic microscope enlarged

J. Basire sc.


C
B
G
A
F
E
D

tinctly without the use of a concave glass, it may be assisted in that respect by a due degree of concavity given to either, or to both the transmitting surfaces of the prism. It is, however, to the upper surface alone that this concavity is given; for since the eye is then situated on the side toward the centre of curvature, it receives all the benefit that is proposed from the periscopic principles.

XIX. *Further Experiments and Observations on the influence of the Brain on the generation of Animal Heat.* *By* B. C. Brodie, *Esq. F. R. S. Communicated to the Society for promoting the knowledge of Animal Chemistry, and by them to the Royal Society.*

Read June 18, 1812.

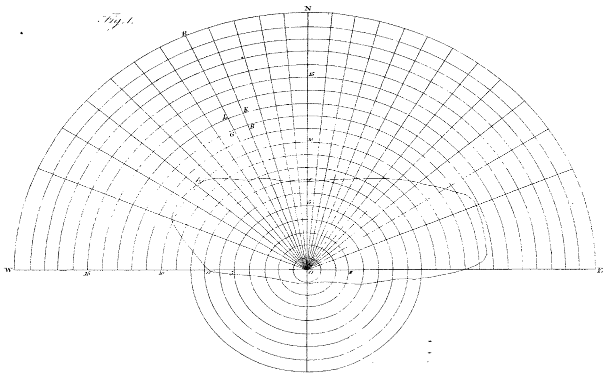
In the Croonian Lecture for the year 1810, I gave an account of some experiments, which led me to conclude that the production of animal heat is very much under the influence of the nervous system. Some circumstances, which I have since met with, illustrate this subject, and seem to confirm the truth of my former conclusions.

In an animal, which is under the influence of a poison, that operates by disturbing the functions of the brain, in proportion as the sensibility becomes impaired, so is the power of generating heat impaired also.

If an animal is apparently dead from a poison of this description, and the circulation of the blood is afterwards maintained by means of artificial respiration, the generation of heat is found to be as completely destroyed, as if the head had been actually removed.

Under these circumstances, if the artificial respiration is kept up until the effects of the poison cease, as the animal recovers his sensibility, so does he also recover the power of generating heat; but it is not till the nervous energy is com-

Fig. 1.



pletely restored, that heat is produced in sufficient quantity to counteract the cold of the surrounding atmosphere.*

In the experiments formerly detailed, as well as in those just mentioned, I observed that the blood underwent the usual alteration of colour in the two systems of capillary vessels, while carbonic acid was evolved from the lungs at each expiration; and hence I was led to believe, that the respiratory function was performed nearly as under ordinary circumstances, and that the usual chemical changes were produced on the blood. It appeared, however, desirable to obtain some more accurate knowledge on this point, and I have therefore instituted a series of experiments, for the purpose of ascertaining the relative quantities of air consumed in breathing, by animals in a natural state, and by animals in which the brain has ceased to perform its office, and I now have the honour of communicating an account of these experiments to this Society.

It has been shewn, by Messrs. Allen and Pepys, first,† that every cubic inch of carbonic acid requires exactly a cubic inch of oxygen gas for its formation; secondly,‡ that when respiration is performed by a warm-blooded animal in atmospheric air, the azote remains unaltered, and the carbonic acid exactly equals, volume for volume, the oxygen gas, which disappears.

There is therefore reason to believe, that the watery vapour, which escapes with the air in expiration, is not formed from

* The poison employed in this experiment should be the essential oil of almonds, or some other, the effects of which speedily subside. If the woorara is employed, so long a time elapses before the poison ceases to exert its influence, that it becomes necessary that the experiment should be made in a high temperature, otherwise the great loss of heat which takes place, is sufficient to prevent recovery.

† Phil. Trans. 1807, Part II.

‡ Phil. Trans. 1808, Part II. Ibid. 1809, Part II.

the union of hydrogen with oxygen in the lungs, but that it is exhaled from the mucous membrane of the mouth and pharynx, resembling the watery exhalation which takes place from the peritonæum, or any other internal surface when exposed; and this conclusion appears to be fully confirmed by the experiments of M. MAGENDIE, lately communicated to the National Institute of Paris.

These circumstances are of importance in the present communication, which they render more simple, as they show, that, in order to ascertain the changes produced on the air in respiration, it is only necessary to find the quantity of carbonic acid given out from the lungs. This becomes an exact measure of the oxygen consumed, and the azote of the air and the watery vapour expired, need not be taken into the account.

For the purpose of examining the changes produced on the air, by animals breathing under the different circumstances abovementioned, I contrived the apparatus, which is represented in the annexed Plate.

Description of the Apparatus.

A. Is a wooden stand, in which is a circular groove $\frac{3}{4}$ of an inch in depth, and the same in width.

B. Is a bell-glass, the rim of which is received in the circular groove of the wooden stand. In the upper part of the bell-glass is an opening, admitting a tube connected with the bladder C.

D. Is a bottle of elastic gum, having a brass stop-cock E connected with it.

F. Is a silver tube, of which one end is adapted to receive the tube of the stop-cock E, while the other extremity, making

a right angle with the rest of the tube, passes through a hole in the wooden stand, and projects into the cavity of the bell-glass, where it makes a second turn also at a right angle, and becomes of a smaller diameter. In the upright part of the tube is an opening G.

The tubes are made perfectly air-tight, where connected with each other, and with the rest of the apparatus, and the circular groove is filled with quicksilver.

The capacity of the bell-glass, allowance being made for the rim, which is received in the groove with the quicksilver, is found to be 502 cubic inches. The capacity of the gum-bottle is 52 cubic inches, and in the calculations after the experiments 2 cubic inches have been allowed for the air contained in the different tubes, and for the small remains of air in the bladder after being nearly emptied by pressure.

Mode of using the Apparatus.

In order to ascertain the quantity of air consumed under ordinary circumstances, the animal was placed on the stand under the bell-glass, the bladder being emptied by pressure, and the gum-bottle being distended with atmospheric air. During the experiment, by pressing occasionally on the gum-bottle, the air was forced from it into the bell-glass. On removing the pressure, the gum-bottle became filled by its own elasticity with air from the bell-glass. Thus the air was kept in a state of agitation, and the dilatation of the bladder prevented the air being forced through the quicksilver under the edge of the bell-glass. At the end of the experiment, the gum-bottle was completely emptied by pressure, and allowed to be again filled with air from the bell-glass: this was repeated

two or three times, and the air in the bottle was then preserved for examination. The proportion of carbonic acid being ascertianed, and the capacities of the different parts of the apparatus, and the space occupied by the animal being known, the total quantity of carbonic acid formed, and consequently of oxygen consumed, was easily estimated.

When the experiment was made on an animal, in whom the functions of the brain were destroyed, and in whom therefore voluntary respiration had ceased, the narrow extremity of the tube was inserted into an artificial opening in the trachea, and the animal being placed under the bell-glass, the lungs were inflated at regular intervals, by means of pressure made on the gum-bottle. The tube being smaller than the trachea, the greater portion of the air in expiration escaped by the side of the tube into the general cavity of the bell-glass, while the gum-bottle filled itself by its own elasticity with air through the opening G. At the end of the experiment, a portion of air was preserved for examination, and the quantity of carbonic acid was estimated in the way already described.

The animals employed in these experiments were of the same species, and nearly of the same size. Attention to these circumstances was judged necessary, that the results might be as conclusive as possible. The chemical examination of the air was made by agitating it in a graduated measure over quicksilver, with a watery solution of potash. My friend Mr. BRANDE gave me his assistance in this part of the present investigation, as he had done on many former occasions. It will be observed, that in estimating the proportion of carbonic acid, no allowance has been made for that contained in

the atmospheric air; first, because the quantity is so small that the omission can occasion no material error; and secondly, because the object is to ascertain, not so much the absolute, as the relative quantities of carbonic acid evolved by animals breathing under different circumstances.

The experiments which I shall first notice, were made on the respiration of animals in a natural state.

Experiment 1. Thermometer 65°, barometer not noted.

A young rabbit was allowed to remain under the bell-glass during 30 minutes. The respired air at the end of this time was found to contain $\frac{1}{20}$ of carbonic acid.

It was ascertained that the rabbit occupied the space of 50 cubic inches.

The capacity of the bell-glass = 502 cubic inches.

That of the gum-bottle 52 cubic inches.

The air in the tubes and bladder = 2 cubic inches.

Then $\frac{502 + 52 + 2 - 50}{20} = \frac{506}{20} = 25.3.$

The rabbit therefore in 30 minutes gave out 25.3 cubic inches of carbonic acid, and consumed the same quantity of oxygen gas, which is at the rate of 50.6 in an hour.

Exp. 2. Thermometer 65°, barometer 30.1 inches.

A somewhat smaller rabbit was allowed to remain under the bell-glass during 30 minutes. The respired air contained $\frac{1}{18}$ of carbonic acid. The animal occupied the space of 48 cubic inches.

$$\frac{502 + 52 + 2 - 48}{18} = \frac{508}{18} = 28.22.$$

The carbonic acid evolved was therefore equal to 28.22 cubic inches in half an hour, which is at the rate of 56.44 cubic inches in an hour.

Exp. 3. Thermometer 64°, barometer 30.2 inches.

A young rabbit, occupying the space of 48 cubic inches, was allowed to remain under the bell-glass, during the same period as in the two former instances. The respired air contained $\frac{1}{18}$ of carbonic acid.

$$\frac{502 + 52 + 2 - 48}{18} = \frac{508}{18} = 28.22.$$

The results of this were therefore precisely the same as those of the last experiment.

These experiments were made with great care. The animals did not appear to suffer any inconvenience from their confinement, and their temperature was unaltered.

The next order of experiments were made for the purpose of ascertaining the quantity of air consumed by animals, in which the circulation of the blood was kept up by means of artificial respiration, after the brain had ceased to perform its functions.

Exp. 4. Thermometer 65°, barometer not noted.

Having procured two rabbits of the same size and colour, I divided the spinal marrow in the upper part of the neck of one of them. An opening was made in the trachea, and the lungs were inflated at first by means of a small pair of bellows. Two ligatures were passed round the neck, one in the upper, and the other in the lower part, behind the trachea. The ligatures were drawn tight, including every thing but the trachea; and the nerves, vessels, and other soft parts between them were divided with a bistoury. Eight minutes after the division of the spinal marrow, the thermometer in the rectum had sunk to 97°. The animal was placed under a bell-glass, and the lungs were inflated by pressing on the gum-bottle about

50 times in a minute. When this process had been continued for 30 minutes, a portion of air was preserved for examination. The heart was found acting regularly, but slowly, the thermometer in the rectum had fallen to 90°.

The second rabbit was killed by dividing the spinal marrow about the same time when the experiment was begun on the first rabbit. Being in the same temperature, the time was noted when the thermometer in the rectum had fallen to 97°, and it was placed under another bell-glass, that it might be as nearly as possible under the same circumstances with the first rabbit. At the end of 30 minutes, the thermometer in the rectum had fallen from 97 to 91.*

The air respired by the first rabbit contained $\frac{1}{25}$ of carbonic acid. The bulk of the rabbit was found = 50 cubic inches.

$$\frac{502 + 52 + 2 - 50}{25} = \frac{506}{25} = 20.24.$$

20.24 cubic inches of carbonic acid were therefore extricated in 30 minutes, which is at the rate of 40.48 cubic inches in an hour.

The carbonic acid given out in the same space of time was less than in the former experiments; but it is to be observed, first, that in consequence of the ligatures the extent of the circulation was diminished; secondly, that in this instance one of the ligatures accidentally slipped, and an ounce of blood was lost in the beginning of the experiment.

As it was desirable to avoid any circumstances, which might occasion a difference in the results, in the subsequent experi-

* In measuring the heat of the rectum in these experiments, care is necessary that the thermometer should always be introduced to exactly the same distance from the external parts, otherwise no positive conclusion can be drawn relative to the loss of heat, as the more internal parts retain their heat longer than the superficial.

ments I employed animals, which had been inoculated with the poison of woorara, or the essential oil of almonds; by which means, while the functions of the brain were completely destroyed, the extent of the circulation was undiminished, and all chance of accidental hæmorrhage was avoided.

Exp. 5. Thermometer 65°, barometer 29.8 inches.

Two rabbits were procured, each occupying the space of 45 cubic inches. They were both inoculated with the woorara poison.

The first rabbit was apparently dead in nine minutes after the application of the poison; but the heart continued to act. The lungs were inflated for about two minutes, by means of a pair of bellows, when the thermometer in the rectum was observed to stand at 98°. The animal was placed under the bell-glass, and artificial respiration was produced by means of pressure on the gum-bottle, as in the last experiment. At the end of 30 minutes, a portion of air was preserved for examination. The thermometer in the rectum had fallen to 91°. The heart still acted with regularity and strength.

The second rabbit died in a few minutes after the inoculation. The time was noted when the thermometer in the rectum had fallen to 98°, and he was placed under a bell-glass. At the end of 30 minutes, the thermometer in the rectum had fallen to 92°.

The air respired by the first rabbit contained $\frac{1}{20}$ of carbonic acid.

$\frac{502 + 52 + 2 - 45}{20} = \frac{511}{20} = 25.55$ cubic inches of carbonic acid evolved in 30 minutes, which is at the rate of 51.1 cubic inches in an hour.

Exp. 6. Thermometer 66°, barometer 30.1 inches.

Two rabbits, each occupying the space of 48 cubic inches, were inoculated with woorara.

In one of them, when apparently dead, the circulation was kept up by means of artificial respiration. He was placed in the apparatus under the bell-glass, and the lungs were inflated from 50 to 60 times in a minute. At this time the thermometer in the rectum stood at 97°. At the end of 35 minutes, a portion of air was preserved for examination. The thermometer had now fallen to 90°. The heart was still acting regularly.

The second rabbit was allowed to lie dead. When the thermometer in the rectum had fallen to 97°, he was placed under another bell-glass. At the end of 35 minutes, the thermometer had fallen to 90°.5.

The air respired by the first rabbit contained $\frac{1}{16}$ of carbonic acid.

$\frac{502 + 2 + 52 - 48}{16} = \frac{508}{16} = 31.75$ cubic inches of carbonic acid evolved in 35 minutes, which is at the rate of 54.43 cubic inches in an hour.

Exp. 7. Thermometer 60°, barometer 30.2 inches.

The experiment was repeated on a rabbit, which had been inoculated with the essential oil of almonds. When he was placed under the bell-glass, the thermometer in the rectum stood at 96°. In a few minutes he gave signs of sensibility, and made efforts to breathe; but as these efforts were at long intervals, the artificial respiration was continued. In half an hour he breathed spontaneously 40 times in a minute. The thermometer in the rectum had fallen to 90°.

The air being examined, was found to contain $\frac{1}{18}$ of carbonic acid.

The rabbit occupied the space of 47 cubic inches.

$\therefore \frac{502+52+2-47}{18} = \frac{509}{18} = 28.275$ cubic inches of carbonic acid evolved in 30 minutes, which is at the rate of 56.55 cubic inches in an hour.

The animal lay as if in a state of profound sleep. At the end of two hours and twenty minutes, from the time of the poison being applied, the thermometer in the rectum had fallen to 79°, and he was again apparently dead; but the heart still continued acting, though feebly, and its action was kept up for 30 minutes longer by means of artificial breathing, when the thermometer had fallen to 76°. The carbonic acid evolved during these last 30 minutes, amounted to nearly 13. cubic inches.

From the precautions with which these experiments were made, I am induced to hope that there can be no material error in their results. They appear to warrant the conclusion, that in an animal in which the brain has ceased to exercise its functions, although respiration continues to be performed, and the circulation of the blood is kept up to the natural standard, although the usual changes in the sensible qualities of the blood take place in the two capillary systems, and the same quantity of carbonic acid is formed as under ordinary circumstances; no heat is generated, and (in consequence of the cold air thrown into the lungs) the animal cools more rapidly than one which is actually dead.

It is a circumstance deserving of notice, that so large a

quantity of air should be consumed by the blood passing through the lungs, when the functions of the brain, and those of the organs dependant on it, are suspended. Perhaps it is not unreasonable to suppose, that by pursuing this line of investigation we may be enabled to arrive at some more precise knowledge respecting the nature of respiration, and the purposes which it answers in the animal economy. It would however be foreign to the plan of the present communication to enter into any speculations on this subject, and I shall therefore only remark, that the influence of the nervous system does not appear to be necessary to the production of the chemical changes, which the blood undergoes in consequence of exposure to the air in the lungs.*

* This conclusion is directly contrary to that deduced by M. DUPUYTREN, from a series of curious experiments, made with a view to ascertain the effects which follow the division of the nerves of the par vagum, and it is an object of some importance in the present investigation, to ascertain in what manner the apparently opposite facts, observed by M. DUPUYTREN and myself, are to be reconciled with each other.

It was observed by this physiologist, that in an animal, in which both the nerves of the par vagum are divided, the blood returned from the lungs has a darker colour than natural, and that the animals, on whom this operation is performed, die sooner or later with symptoms of asphyxia, notwithstanding the air continues to enter the lungs; and hence he concludes, that the changes which are produced on the blood in respiration are not the result of a mere chemical process, but are dependent on the nervous influence, and cease to take place when the communication between the lungs and the brain is destroyed.

M. PROVENÇAL, in prosecuting this inquiry, ascertained that the animals subjected to this experiment give out less carbonic acid than before.

M. BLAINVILLE observed, that the frequency of the inspirations is much diminished; and M. DUMAS restored the scarlet colour of the arterial blood by artificially inflating the lungs, and from these and other circumstances, he has arrived at conclusions very different from those of M. DUPUYTREN.

My own observations exactly correspond with those of M M. DUMAS and BLAINVILLE. After the nerves of the par vagum are divided, a less quantity of carbonic

The facts now, as well as those formerly adduced, go far towards proving, that the temperature of warm-blooded animals is considerably under the influence of the nervous system; but what is the nature of the connection between them? whether is the brain directly or indirectly necessary to the production of heat? these are questions to which no answers can be given, except such as are purely hypothetical. At present we must be content with the knowledge of the insulated fact: future observations may, perhaps, enable us to refer it to some more general principle.

We have evidence, that, when the brain ceases to exercise its functions, although those of the heart and lungs continue to be performed, the animal loses the power of generating heat. It would, however, be absurd to argue from this fact, that the chemical changes of the blood in the lungs are in no way necessary to the production of heat, since we know of no instance in which it continues to take place after respiration has ceased.

It must be owned, that this part of physiology still presents an ample field for investigation.

Of opinions sanctioned by the names of BLACK, LAPLACE, LAVOISIER, and CRAWFORD, it is proper to speak with caution

acid is evolved, the inspirations are much diminished in frequency, and the blood in the arteries of the general system assumes a darker hue; but its natural colour may be restored by artificially inflating the lungs, so as to furnish a greater supply of air to the blood circulating through them.

We may suppose, that, on the division of these nerves, the sensibility of the lungs is either extremely impaired, or altogether destroyed, so that the animal does not feel the same desire to draw in fresh air: in consequence his inspirations become less frequent than natural, and hence arise the phenomena produced by this operation.

and respect, but without trespassing on these feelings, I may be allowed to say, that it does not appear to me that any of the theories hitherto proposed afford a very satisfactory explanation of the source of animal heat.

Where so many and such various chemical processes are going on, as in the living body, are we justified in selecting any one of these for the purpose of explaining the production of heat?

To the original theory of Dr. Black, there is this unanswerable objection, that the temperature of the lungs is not greater than that of the rest of the system. To this objection, the ingenious and beautiful theory of Dr. Crawford is not open; but still it is founded on the same basis with that of Dr. Black, "the conversion of oxygen into carbonic acid in the lungs," and hence it appears to be difficult to reconcile either of them with the results of the experiments which have been related.

It may perhaps be urged, that as in these experiments the secretions had nearly, if not entirely ceased, it is probable that the other changes, which take place in the capillary vessels had ceased also, and that although the action of the air on the blood might have been the same as under ordinary circumstances, there might not have been the same alteration in the specific heat of this fluid, as it flowed from the arteries into the veins. But, on this supposition, if the theory of Dr. Crawford be admitted as correct, there must have been a gradual, but enormous accumulation of latent heat in the blood, which we cannot suppose to have taken place without its nature having been entirely altered. If the blood undergoes the usual change in the capillary system of the pulmonary, it is

probable that it must undergo the usual change in the capillary system of the greater circulation also, since these changes are obviously dependent on and connected with each other. The blood in the aorta and pulmonary veins was not more florid, and that in the vena cava and pulmonary artery was not less dark-coloured than under ordinary circumstances. We may moreover remark, that the most copious secretions in the whole body are those of the insensible perspiration from the skin, and of the watery vapour from the mouth and fauces, and the effect of these must be to lower rather than to raise the animal temperature. Under other circumstances also the diminution of the secretions is not observed to be attended with a diminution of heat. On the contrary, in the hot fit of a fever, when the scanty dark-coloured urine, dry skin, and parched mouth indicate that scarcely any secretions are taking place, the temperature of the body is raised above the natural standard, to which it falls when the constitution returns to its natural state, and the secretions are restored.

It has been observed, by a distinguished chemist, that "the experiments to determine the specific heat of the blood are of so very delicate a nature, that it is difficult to receive them with perfect confidence."* The experiments of Dr. CRAWFORD for this purpose were necessarily made on blood out of the body, and at rest. Now, when blood is taken from the vessels, it immediately undergoes a remarkable chemical change, separating into a solid and a fluid part. This separation is not complete for some time; but whoever takes the pains to make observations on the subject, can hardly doubt that it begins to take place immediately on the blood being drawn. Can expe-

* THOMSON's History of the Royal Society, p. 129.

riments on the blood, under these circumstances, lead to any very satisfactory conclusions, respecting the specific heat of blood circulating in the vessels of the body? The diluting the blood with large quantities of water, as proposed by Dr. CRAWFORD, does not altogether remove the objection, for this only retards, it does not prevent coagulation, and some time must, at any rate, elapse, while the blood is flowing and the quantity is being measured, during which the separation of its solid and fluid parts will have begun to take place.

More might be said on this subject; but I feel anxious to avoid, as much as possible, controversial discussion. It is my wish not to advance opinions, but simply to state some facts, which I have met with in the course of my physiological investigations. These facts, I am willing to hope, possess some value; and they may perhaps lead to the developement of other facts of much greater importance. Physiology is yet in its infant state. It embraces a great number and variety of phenomena, and of these it is very difficult to obtain an accurate and satisfactory knowledge; but it is not unreasonable to expect, that by the successive labours of individuals, and the faithful register of their observations, it may at last be enabled to assume the form of a more perfect science.

XX. *On the different Structures and Situations of the Solvent Glands in the digestive Organs of Birds, according to the nature of their Food and particular Modes of Life. By* Everard Home, *Esq. F. R. S.*

Read June 18, 1812.

THE solvent glands in birds are larger and more distinct from the other parts of the digestive organs than in the class mammalia, which has enabled me to ascertain many circumstances respecting their structure, not to be determined by examining the stomachs of quadrupeds. An account of these is contained in the present paper.

To make the following descriptions more clear and distinct, I shall divide the digestive organs of birds, whether they live on animal or vegetable food, into four parts. The first, is the dilatation of the œsophagus, which forms a reservoir for the food, and which is called the crop. The second, is the part into which the ducts of the solvent glands open, which, I shall call the cardiac cavity. The third, is the cavity embraced by the digastric muscle, or gizzard. The fourth, is the space between the opening of the gizzard, and beginning of the duodenum, which I shall call the pyloric cavity, although in some instances it appears scarcely to deserve that name.

The solvent glands in the whole of the extensive genus falco of LINNÆUS, are cylindrical bodies with very small canals, a villous internal surface, and thick coats, open at one end, closed,

and rounded off at the other; they are placed on the outside of the membrane which lines the cardiac cavity, they lie parallel to one another, and nearly at right angles to the membrane through which they open, the closed end being slightly turned upwards, so as to make the orifice the most depending part. In the golden eagle, (the Falco Chrysaëtos *Linn.*) and the sea eagle, (Falco ossifragus), they form, altogether, a broad compact belt: but in the hawk, (Falco Nisus) this belt is slightly divided into four distinct portions; immediately below these glands the cavity becomes wider, and is inclosed in a digastric muscle of weak power, with a flat tendon on each side. The internal surface of this cavity, which is the gizzard, is soft and vascular.*

In all birds that live on animal food, the solvent glands appear to have a similar structure to that which has been just described, only differing in size and situation. The following are the most material differences which I have met with respecting their situation.

In the Soland goose, (Pelecanus bassanus,) these glands are rather larger than in the eagle, but are placed in the dilated part of the cavity of the gizzard, forming a complete belt of great breadth, consequently are extremely numerous. Their situation and appearance is shewn in the annexed drawing.

In the heron, (Ardea cinerea), the solvent glands are in the same situation as in the Soland goose; they are thinly scattered, and do not form a complete belt, being more numerous on the anterior and posterior surfaces. A ball of

* An engraving of this appearance is given in the Philosophical Transactions for the year 1807, p. 178, pl. xii.

fish bones held together by mucus, was found in the cavity of the gizzard.

In the cormorant, (Pelecanus Carbo), the situation of the solvent glands is the same as in the Soland goose, but they only form two circular spots, one anterior, the other posterior.*

In all these birds, the inner membrane of the gizzard is soft and smooth, but that portion which covers the solvent glands, has a more spongy or villous appearance; and this part is found to secrete a mucus which the other parts do not. This fact appears to be ascertained by the following circumstances: on examining the gizzard of a cormorant that died in consequence of an inflammation in the œsophagus, which had been communicated to the internal membrane of the gizzard, a viscid mucus was found upon the surface covering the solvent glands, and this was not met with in any other part, so that the mucus had been evidently secreted there, and was afterwards coagulated by the liquor of the solvent glands poured upon it, coagulation being the first process which takes place in the act of digestion. This explains the circumstance of ascarides being frequently found enveloped in mucus in this part of the cormorant's gizzard, the mucus on which they feed being secreted in consequence of the irritation they produce on the membrane. In the same manner the flukes in the biliary ducts of the sheep, increase the secretion of the bile by irritating these canals, and then feed on it.

It is generally believed that mucus is secreted by surfaces as well as by glandular structure, but I know of no evidence that

* An engraving of this gizzard is given in the Phil. Trans. for the year 1807, pl. x. p. 178.

the mucus of the stomach comes under this description, except what is now stated.

In birds that live upon fish, and sea insects with crustaceous coverings, as the sea gull, (Larus canus), the gizzard has a horny cuticular lining, and the solvent glands are in the same situation as in the genus falco. In the gizzard were found the scales of small fishes. The appearance of the solvent glands and gizzard are shewn in the annexed engraving.

In those birds that live on land insects, some of whose coverings are soft, others hard, there is a difference in the structure of the digestive organs from what has been described. The solvent glands are placed in a triangular form in the cardiac cavity, which is very large, and immediately under it is a small gizzard with a horny lining. Of this kind is the woodpecker, (Picus minor). A representation of the parts in this bird is annexed.

There is still another variety in the structure of these organs.

In the little auk, the (Alca Alle), the solvent glands are spread over a greater extent of surface than in any other bird that lives on animal food, and the form of the digestive organs is peculiar to itself. The cardiac cavity appears to be a direct continuation of the œsophagus, distinguished from it by the termination of the cuticular lining, and the appearance of the solvent glands. This cavity is continued down with very gradual enlargement below the liver, and is then bent up to the right side, and terminates in a gizzard: when the cavity is laid open, the solvent glands are seen at its upper part, every where surrounding it, but lower down they lie principally on the posterior surface, and where it is bent upwards towards

the right side, they are entirely wanting. The gizzard has a portion of its anterior and posterior surfaces opposite each other, covered with a horny cuticle. These appearances are shewn in the annexed drawing.

This peculiar formation of the digestive organs of the little auk, appears to be fitted for economizing the food; which may be rendered more necessary in a bird, that spends a portion of the year in the frozen regions of the North, where supplies of nourishment must be very precarious.

In birds that live principally on vegetable food, the solvent glands have a different structure, according to the substances the birds are intended to feed upon, and vary a good deal in situation, according to the habits of life. The following are the most remarkable instances of such difference, both with respect to structure and situation.

In the pigeon (Columba domestica), their situation is the same as in the genus falco, but their size is small, the external orifices large, and the coats thin, so that they resemble the glands in the English heron, but having larger cavities.

In the swan, the (Anas Cygnus), the solvent glands appear to be cylinders, as in the genus falco, but are not straight, bending upon one another in a direction obliquely upwards; their internal surface is not villous as in the genus falco, but rather broken and irregular.

In the goose (Anas Anser), the solvent glands have the same situation as in the swan, and resemble them in their external appearance, but when laid open the sides are found to be cellular.

In the common fowl (Phasianus Gallus), these glands are

made up of four small short processes uniting in a middle tube, which opens externally by one orifice.

In the turkey (the Meleagris Gallipavo), the solvent glands consist of four small processes, which diverge from one another in opposite directions.

In a species of parrot, (the Psittacus æstivus), the cardiac cavity is unusually large and long, and the solvent glands are spread over a considerable portion of its surface, the gizzard is very small. Its appearance is represented in the annexed engraving.

In many large birds that only walk and run, their wings being too small to enable them to fly, the digestive organs are formed in many respects differently from those of other birds.

In the cassowary, (Casuarius Emeu), the solvent glands are situated between the crop and gizzard, as in many other birds, but this part is dilated into a large cavity, and separated from the gizzard by an oblique muscular valve; in this cavity the food may be retained for some time, but cannot be triturated there, since the stones and other hard bodies swallowed, will readily force a passage into the gizzard.

I have not had an opportunity of examining the solvent glands in the cassowary, and therefore can say nothing respecting their structure from my own observation.

In the American ostrich, (the Rhea americana), the solvent glands are fewer in number than in other birds. They only occupy a small portion of a circular form, on the posterior side of the cardiac cavity; this however is compensated by the complex structure of which they are composed. To each gland there is one common orifice; when the cavity to which it leads is laid open, three smaller orifices are exposed,

each of which communicates with five or six processes like the fingers of a glove. The structure is similar to that of the solvent glands of the beaver, among quadrupeds.

The cardiac cavity, in which the solvent glands are situated, is dilated to a large size, as in the cassowary, and there is a similar muscular valve separating it from the gizzard. The digastric muscle is weak; but the fibres of which it is composed, and the tendons between the two bellies of the muscle, are beautifully distinct. The appearance of these parts is shewn in the annexed drawing.

In the African ostrich, (the Struthio Camelus,) the solvent glands are unusually numerous, similar in structure to those of the American species; the space in which they are situated is not only dilated into a cavity, but is continued down below the liver, and then bent up upon itself towards the right side, where it terminates in a strong gizzard nearly at the same height as the beginning of the cardiac cavity.

This cardiac cavity of such unusual length and uncommon form is lined with a strong cuticle, except upon the left side where the solvent glands are placed, extending from the top to the bottom, and about four inches in breadth.

The gizzard is unusually small; the grinding surfaces do not admit of being separated to a great distance from one another, and on one side there are two grooves, and two corresponding ridges on the other. Beyond the cavity of the gizzard is an oval aperture with six ridges covered with cuticle to prevent any thing passing out of the gizzard till it is reduced to a small form. These appearances are shewn in the annexed drawing.

In this bird the reverse takes place of what was mentioned to happen in the cassowary and American ostrich, for the stones and other hard bodies swallowed by these birds, must, from their weight, force a way into the gizzard, which is a large cavity adapted to receive them; but here all such substances must remain in the cardiac cavity, both from its being the most depending part, and from the cavity of the gizzard being too small to admit of their entering it.

The cardiac cavity contained stones of various sizes, pieces of iron, and halfpence; but between the grinding surfaces of the gizzard, there were only broken glass beads of different colours, and hard gravel mixed with the food.

In taking a review of the structure of the digestive organs of the cassowary, the American, and African ostrich, whose mode of progressive motion is the same, we find their organs very differently circumstanced with respect to the means of œconomising their food.

The cassowary and American ostrich differ from birds that fly, in having the solvent glands placed in a cavity of unusual size and the muscular structure of the gizzard uncommonly weak; the mode of progressive motion, which is a kind of run, producing so much agitation between the stones and the food, as to render a stronger muscular action unnecessary.

In the cassowary there appear to be no considerations of œconomy in the management of the food in the process of digestion, the solvent glands are less complex than in the ostrich, as is avowed by those who have examined them,* the food has a free passage from the gizzard into the intestines, which are unusually wide and short, so that its passage

* Vide PERRAULT's Comp. Anat. 1676.

through them is very rapid, and is rendered still more so by the stones of a large size employed in the gizzard passing out at the anus. This I learnt from Sir JOSEPH BANKS, who was present at the Cape of Good Hope when one of these birds, to his great astonishment, voided nearly half a bucket full of stones.

In the American ostrich there will be less waste of the food than in the cassowary, as the solvent glands are of a more complex structure, as there is a less ready outlet from the gizzard, and as the intestines are longer and have a variety of convolutions.

In the African ostrich the means of œconomising the food are greater than in other birds; the glands have the same structure as in the American species, are more numerous, are spread over a larger surface, there is a more extensive cavity in which the substances it feeds upon are triturated; and beyond this, a grooved gizzard for the more accurate breaking down of the food. The intestines also are longer and more varied in their course.

All these provisions of nature fit this bird to live in the sandy deserts, of which it is the natural inhabitant; and are not bestowed upon the others that live in countries where food is more abundant.

It is a curious circumstance that the situation of the solvent glands, the shape of the cardiac cavity and position of the gizzard in the alca alle among carnivorous birds is nearly the same as that of the African ostrich among birds that live principally on vegetable food.

EXPLANATION OF THE PLATES.

Plate XI.

The gizzard of the Soland goose laid open to show the situation and appearance of the solvent glands. The engraving is of the natural size.

Plate XII.

Shows the form and internal structure of the cardiac cavity and gizzard in the wood-pecker, the sea-gull, and the little auk, all of the natural size.

Fig. 1. The external appearance of the cardiac cavity and gizzard in the wood-pecker.

2. The internal appearance of the cardiac cavity and gizzard of the wood-pecker.

3. The internal appearance of the cardiac cavity and the cavity of the gizzard in the sea-gull.

4. The internal appearance of the cardiac cavity and gizzard of the little auk.

Plate XIII.

Two views of the digestive organs of the parrot of the natural size.

Fig. 1. An external view of the crop, cardiac cavity, and gizzard.

2. An internal view of the same parts.

Plate XIV.

An external view of the cardiac cavity and gizzard of the American ostrich, one-fourth of the natural size.

PLATE XV.

An internal view of the cardiac cavity and gizzard of the American ostrich, on the same scale as the last plate.

The orifices of the solvent glands are very conspicuous in the cardiac cavity.

PLATE XVI.

An external view of the cardiac cavity and gizzard of the African ostrich, one-sixth of the natural size.

PLATE XVII.

Represents the internal view of the African ostrich's gizzard, and also a series of solvent glands belonging to different birds.

Fig. 1. The internal surface of the cardiac cavity and gizzard of the African ostrich. On the same scale as in the last plate.

Fig. 2. A series of solvent glands to show the different appearances which they put on.

W. Clift del.

J. Basire sc.

Fig. 1.

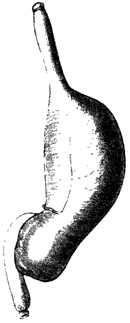


Fig. 2.

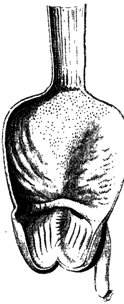


Fig. 3

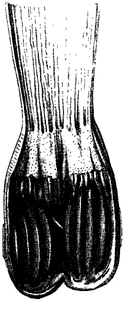
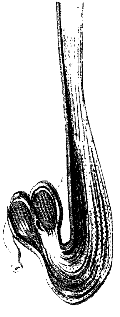


Fig. 4



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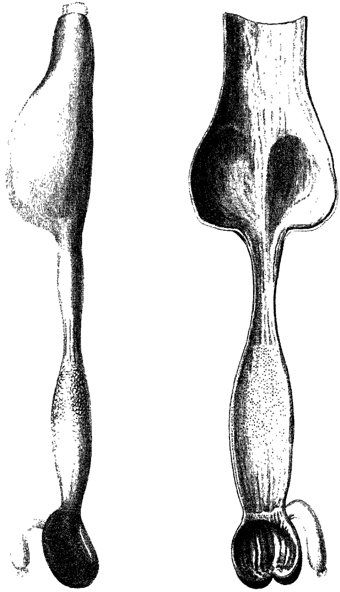

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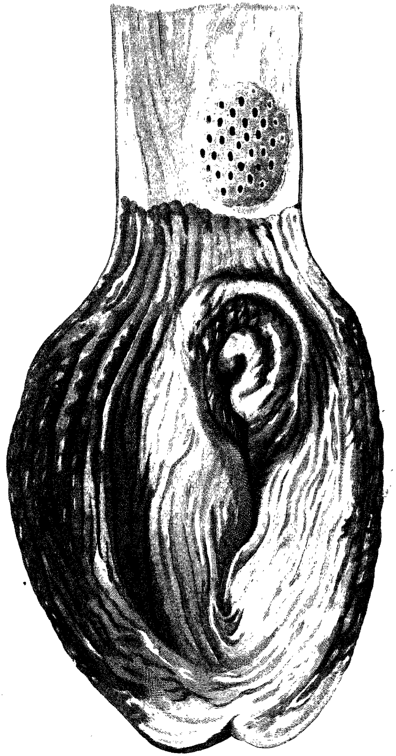

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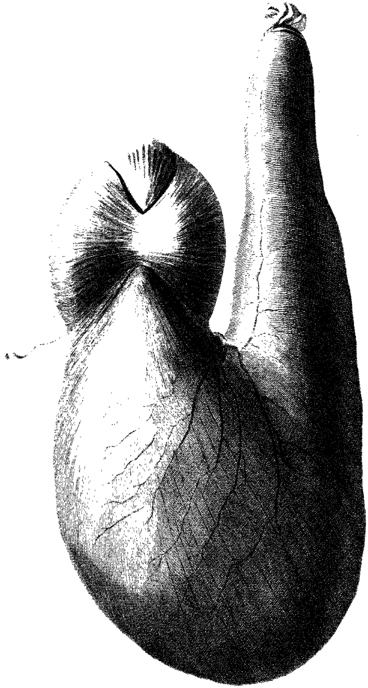
B. Basire sc.



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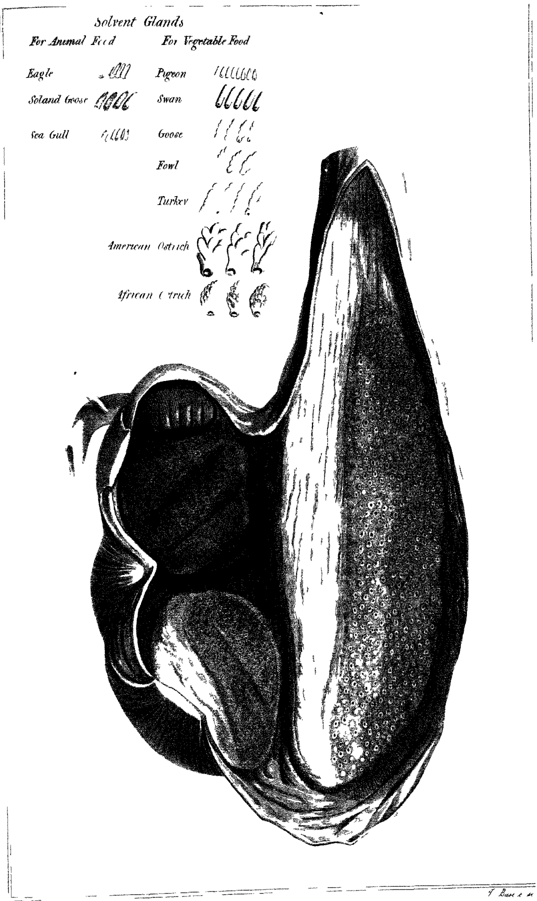

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XXII. *On some Combinations of Phosphorus and Sulphur, and on some other Subjects of Chemical Inquiry. By Sir* Humphry Davy, *Knt. LL. D. Sec. R. S.*

Read June 18, 1812.

1. *Introduction.*

In this paper, I shall do myself the honour of laying before the Society, the results of some experiments on phosphorus and sulphur, which establish the existence of some new compounds, and which offer decided evidences in favour of an idea that has been for some time prevalent amongst many enlightened chemists, and which I have defended in former papers published in the Philosophical Transactions; namely, that bodies unite in definite proportions, and that there is a relation between the quantities in which the same element unites with different elements.

I shall not enter into a minute detail of the methods of experimenting that I employed; I shall confine myself to general statements of the facts. The common manipulations of chemistry are now too well known to require any new illustrations: and to dwell upon familiar operations, would be to occupy unnecessarily and tediously the time of this learned body.

2. *Of some Combinations of Phosphorus.*

In a paper read before the Royal Society in 1810, I have described the mutual action of phosphorus and oxymuriatic gas, or chlorine. I have noticed two compounds which appear to be distinct and peculiar bodies, formed by the union of the gas and the inflammable substance. One is solid, white, and crystalline in its appearance; easily volatile, and capable of forming a fixed infusible substance by uniting with ammonia. The other is fluid, limpid as water, and, as I have since found, of specific gravity 1.45; it produces dense fumes by acting upon the water of the atmosphere, and when exposed to the atmosphere gradually disappears, leaving no residuum.

The composition of the white sublimate is very easily ascertained by synthetical experiments, such as I have described on a former occasion in the Transactions. By employing chlorine dried by muriate of lime, in great excess, and making the experiments in exhausted vessels, and admitting solution of chlorine to ascertain the quantity of gas absorbed, I have ascertained that 3 grains of phosphorus unite with about 20 grains of chlorine to form the sublimate.

If the phosphorus be in great excess in the experiment of its combustion in chlorine, some of the liquor is formed with the sublimate; but to obtain it in considerable quantities, phosphorus should be passed in vapour through heated powdered corrosive sublimate. A bent glass tube may be used for the process, and the liquor condensed in a cold vessel connected with the tube.

I have not been able to determine its composition by synthetical experiments; but by pouring it gradually into water,

suffering the water to become cool after each addition of the liquor, and then precipitating the solution by solution of nitrate of silver, I have ascertained the quantity of chlorine and of phosphorus it contains. 13.6 grains, treated in this way, afforded 43 grains of horn silver.

It is evident from this analysis, compared with the result of the synthetical experiments on the sublimate, that the quantity of phosphorus being the same, the sublimate contains double as much chlorine as the liquor.

When phosphorus is heated in the liquor, a portion is dissolved, and it then when exposed to the atmosphere leaves a film of phosphorus, which when the liquor is thrown on paper usually inflames: a substance of this kind was first procured by M M. GAY LUSSAC and THENARD, by distilling phophorus and calomel together; and it may be produced in the experiment with corrosive sublimate, if sufficient heat be used to sublime the phosphorus, or if there be not an excess of the corrosive sublimate. I have made no experiments in order to ascertain the quantity of phosphorus the liquor will dissolve.

When the white sublimate is made to act upon water, it dissolves in it producing much heat. The solution evaporated affords a thick liquid, which is a solution of pure phosphoric acid, or a hydrat of phosphoric acid.

When the liquor is treated with water in the same way, it furnishes likewise a thick fluid of the consistence of syrup, which crystallizes slowly by cooling, and forms transparent parallelopipedons.

This substance has very singular properties: when it is heated pretty strongly in the air, it takes fire and burns brilliantly, emitting at the same time globules of gas, that inflame

at the surface of the liquid. This substance may be called *hydrophosphorous* acid; for it consists of pure phosphorous acid and water. This is proved by the action of ammoniacal gas upon it; when it is heated in contact with ammonia, water is expelled and phosphite of ammonia formed; and it is likewise shewn by the results of its decomposition in close vessels, which are phosphoric acid and a peculiar compound of phosphorus and hydrogen.

Ten parts in weight of the crystalline acid I found produced about 8.5 parts of solid phosphoric acid, and the elastic product must of course have formed the remainder of the weight, allowing for a small quantity of the substance not decomposed.

The peculiar gas is not spontaneously inflammable; but explodes when mixed with air, and heated to a temperature rather below 212°.

Its specific gravity appeared from an experiment in which a small quantity of it only was weighed, to be to that of air nearly as 87 to 100. Water absorbed about one-eighth of its volume of this gas. Its smell was disagreeable, but not nearly so fetid as that of common phosphuretted hydrogene.

When it was detonated with oxygene, it was found that three of it in volume absorbed more than five in volume of oxygene, and a little phosphorus was precipitated.

When potassium was heated in contact with it, its volume increased rapidly till it became double, and then no further effect was produced. The potassium was partly converted into a substance having all the characters of phosphuret of potassium; and the residual gas absorbed the same quantity of oxygene by detonation as pure hydrogene. When sulphur

was sublimed in the gas over mercury, the volume was likewise doubled; a compound of phosphorus and sulphur was formed, and the elastic fluid produced had all the characters of sulphuretted hydrogene.

It appears from these experiments, that the peculiar gas consists of 4.5 of hydrogene in weight to 22.5 phosphorus; and its composition being known, it is easy to determine the composition of the hydrophosphorous acid, and likewise the quantity of oxygene required by a given quantity of phosphorous acid to be converted into phosphoric acid; for, for every volume of gas disengaged, a volume of oxygene must have been fixed in the phosphoric acid.

And calculating for 174 grains, 30 parts of oxygene must be fixed in the 150 parts of phosphoric acid, and 20 parts of phosphorus disengaged in combination with 4 parts of hydrogene; and on the idea of representing the proportions in which bodies combine by numbers, if hydrogene be considered as unity and water as composed of two proportions of hydrogene, 2, and one of oxygene 15,* phosphorus will be represented by 20.

When the compounds of chlorine and phosphorus are acted on by a small quantity of water, muriatic acid gas is disengaged with violent ebullition, the water is decomposed, and it is evident that for every volume of hydrogene disengaged in combination with the chlorine, half a volume of oxygene must be combined with the phosphorus; and the products of the mu-

* Supposing 100 cubical inches of the gas to weigh 27 grains.——27 — 4.5 the weight of 200 cubical inches of hydrogene = 22.5 grains.

† This mode of estimation is the same as that I have adopted on a former occasion, except that the number representing oxygene is doubled to avoid a fractional part.

tual decomposition of water, and the phosphoric compounds of chlorine are merely the phosphoric acid from the sublimate and the phosphorous acid from the liquor, and muriatic acid gas; so that the quantity of phosphorus being the same, it is evident that phosphoric acid must contain twice as much oxygene as phosphorous acid, which harmonizes with the results of the decomposition of hydrophosphorous acid. For supposing water to be composed of two proportions of hydrogene, and one of oxygene, and the number representing it 17; then 174 parts of hydrophosphorous acid must consist of two proportions; 34 parts of water, and four proportions of phosphorous acid, containing 80 of phosphorus and 60 of oxygene; and three proportions of phosphoric acid must be formed, containing three proportions of phosphorus 60, and six proportions of oxygene 90, making 150.

It is scarcely possible to imagine more perfect demonstrations of the laws of definite combination, than those furnished in the mutual action of water and the phosphoric compounds. No products are formed except the new combinations; neither oxygene, hydrogene, chlorine, nor phosphorus is disengaged, and therefore the ratio in which any two of them combine being known, the ratios in which the rest combine, in these cases, may be determined by calculation.

I converted phosphorus into phosphoric acid, by burning it in a great excess of oxygene gas over mercury in a curved glass tube, and heated the product strongly. I found in several processes of this kind, that for every grain of phosphorus consumed, four cubical inches and a half of oxygene gas were absorbed; which gives phosphoric acid as composed of 20 of phosphorus to 30.6 of oxygene; a result as near as can be expected to the

results of the experiments on the sublimate and the hydrophosphorous acid.

Unless the product of the combustion of phosphorus is strongly heated in oxygene, the quantity of oxygene absorbed is less, so that it is probable that phosphorous acid is formed, as well as phosphoric acid.

Phosphorous acid is usually described, in chemical authors, as a fluid body, and as formed by the slow combustion of phosphorus in the air; but the liquid so procured is, I find, a solution of a mixture of phosphorous and phosphoric acids. And the vapour arising from phosphorus in the air at common temperatures, is a combination of phosphorous acid and the aqueous vapour in the air, and is not, I find, perceived in air artificially dried.

In this case, the phosphorus becomes covered with a white film, which appears to be pure phosphorous acid, and it soon ceases to shine.

A solid acid, volatile at a moderate degree of heat, may be produced by burning phosphorus in very rare air, and this seems to be phosphorous acid free from water; but some phosphoric acid, and some yellow oxide of phosphorus, are always formed at the same time.

The peculiar gas differs exceedingly from phosphoretted hydrogene formed by the action of earths and alkalies and phosphorus upon water; for this last gas is spontaneously inflammable, and its specific gravity is seldom more than half as great, and it does not afford more than 1.5 its volume of hydrogene when decomposed by potassium; it differs in its qualities in different cases, and probably consists of different mixtures of hydrogene with a peculiar gas, consisting of 2 parts of hydrogene and

20 of phosphorus; or it must contain several proportions of hydrogene to one of phosphorus.

I venture to propose the name *hydrophosphoric* gas for the new gas; and according to the principles of nomenclature, I have proposed in the last Bakerian lecture, the liquor containing 20 of phosphorus to 67 of chlorine may be called *phosphorane*, and the sublimate *phosphorana.*

3. *Of some Combinations of Sulphur.*

I have shewn, in a paper published in the Philosophical Transactions for 1810, that sulphuretted hydrogene is formed by the solution of sulphur in hydrogene, and I have supposed that sulphureous acid, in like manner, is constituted by a solution of sulphur in oxygene. There is always a little condensation of volume in experiments on the combustion of sulphur in oxygene; but this may fairly be attributed to some hydrogene loosely combined in the sulphur; and to the production of a little sulphuric acid by the mutual action of hydrogene, oxygene, and sulphur.

It is only necessary, if these data be allowed, to know the difference between the specific gravity of sulphureous acid gas and oxygene, and sulphuretted hydrogene and hydrogene, to determine their composition.

In the Philosophical Transactions for 1810, page 254, I have somewhat under-rated the weights of sulphuretted hydrogene and sulphureous acid gasses: for I have since found, that the cubical inch measures, employed for ascertaining the volumes of gas weighed, were not correct. From experiments which I think may be depended upon, as the weights of the gasses were merely compared with those of equal volumes of common air, I found that 100 cubical inches of sulphureous acid gas weighed 68 grains

at mean temperature and pressure, and 100 cubical inches of sulphuretted hydrogene 36.5 grains, and the last result agrees very nearly with one given by MM. GAY LUSSAC and THENARD, and one gained by my brother Mr. JOHN DAVY.

If 34, the weight of 100 cubical inches of oxygene gas, be subtracted from 68, it will appear that sulphureous acid consists of equal weights of sulphur and oxygene, an estimation which agrees very nearly with one given by M. BERZELIUS; and if 2.27, the weight of 100 cubical inches of hydrogene be subtracted from 36.5, the remainder 34.23 will be the quantity of sulphur in the gas; and the number representing sulphur may be stated as 30; and sulphureous acid as composed of one proportion of sulphur 30, and two of oxygene 30; and sulphuretted hydrogene as composed of one proportion of sulphur, and two of hydrogene.

From the experiments of MM. GAY LUSSAC, it appears that sulphuric acid decomposed by heat affords one volume of oxygene to two of sulphureous acid: from this it would appear to be composed of one proportion of sulphur to three of oxygene. I have endeavoured, in several trials by common heat and by electricity, to combine sulphureous acid gas with oxygene, so as to form a sulphuric acid free from water, but without success; and it is probable, that three proportions of oxygene cannot be combined with one proportion of sulphur, except by the intermedium of water. Mr. DALTON has supposed, that there is a solid sulphuric acid formed by the action of sulphureous acid gas upon nitrous acid gas. But I find, that when dried sulphureous acid gas and nitrous acid gas are mixed together, there is no action; but by introducing the vapour of water, they form together a solid crystalline hydrat; which when thrown into

water gives off nitrous gas, and forms a solution of sulphuric acid.

I have referred, in the Philosophical Transactions, to the combination of chlorine and sulphur. I have been able to form no compound of these bodies, which does not deposit sulphur by the action of water. When sulphur is saturated with chlorine, as in Dr. THOMSON's sulphuretted liquor, it appears to contain, from my experiments, only 67 of chlorine to 30 of sulphur.

4. *Some general Observations.*

It is a fact worthy of notice, that phosphoric and sulphuric acids should contain the same quantity of oxygene to the same quantity of inflammable matter; and yet that the oxygene should be combined in them, with such different degrees of affinity. Phosphorous acid has a great tendency to unite with oxygene, and absorbs it even from water: and sulphureous acid can only retain it when water is present.

The relation of water to the composition of many bodies has already occupied the attention of some distinguished chemists, and is well worthy of being further studied; most of the substances obtained by precipitation from aqueous solutions are, I find, compounds of water.

Thus zircona, magnesia, silica, when precipitated and dried at 212° still contain definite proportions of water. And many of the substances which have been considered as metallic oxides, that I have examined, obtained from solutions, agree in this respect; and their colours and other properties are materially influenced by this combined water.

I shall give an instance. The substance which has been called the white oxide of manganese is a compound of water and the

protoxide of manganese, and when heated strongly, it gives off its water and becomes a dark olive oxide.

It has been often suspected, that the contraction of volume produced in the pure earths by heat, is owing to the expulsion of water combined with them. The following fact seems to confirm this suspicion, and offers a curious phenomenon.

Zircona, precipitated from its solution in muriatic acid by an alkali, and dried at a temperature below 300°, appears as a white powder, so soft as not to scratch glass. When heated to 700° or 800°, water is suddenly expelled from it, and notwithstanding the quantity of vapour formed, it becomes at the moment red hot. After the process, it is found harsh to the feel, has gained a tint of gray, its parts cohere together, and it is become so hard as to scratch quartz.

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